The invariant Hilbert scheme and the Cox realization of nilpotent orbit closures in $\mathfrak{sl}_n(\mathbb{C})$.

§. The Hilbert scheme.

① The Grothendieck-Hilbert scheme. X : a smooth quasi-proj. scheme of finite type.

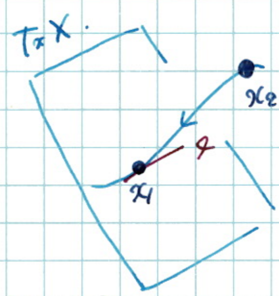
$\mathrm{Hilb}^P(X)$: a parameter space of $Z \subset X$ (closed subscheme) with Hilbert polynomial P.

· $P \equiv n$ ($n \in \mathbb{Z}_{>0}$). constant.

$\mathrm{Hilb}^n(X)$. $Z \subset X$ 0-dimensional cl. subscheme of length n.
"the punctual Hilbert scheme" i.e. $\dim_{\mathbb{C}} \mathcal{O}(Z) = n$.

e.g. $\dim X = 2$, $n = 2$.

$\mathrm{Hilb}^2(X)$. Hilbert scheme of two points on X.



two distinct points x_1, x_2.
double points $x \cdot (x_2 \to x_1)$.

$$\mathrm{Hilb}^2(X) = \underbrace{\{ x_1, x_2 \mid x_1 \neq x_2 \}}_{\text{general points.}} \cup \underbrace{\{ x, \ell \mid \ell \in T_x X \}}_{\text{limit \& tangent direction.}}$$

$$\begin{array}{ccc} \mathrm{Hilb}^n(X) & \longrightarrow & \mathrm{Sym}^n(X) := X^n/\mathfrak{S}_n \\ Z & \longmapsto & \sum_{x \in X} \dim_{\mathbb{C}} \mathcal{O}_{Z,x} \cdot x \end{array}$$

"the Hilbert-Chow morphism."

Action : permutation of the coordinates.

Thm (Fogarty '68).
$\dim X = 2$. $\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X)$ is a resolution of singularities. (smooth of dim. $2n$.)

②. The G-Hilbert scheme. $G \subset SL_n$ (finite subgroup), $G \curvearrowright \mathbb{C}^n$.

$\mathbb{C}^n/G$: the space of G-orbits in $\mathbb{C}^n$.

$G\text{-}\mathrm{Hilb}(\mathbb{C}^n) \longrightarrow \mathbb{C}^n/G$. "the Hilbert-Chow morphism"

general points.
= free G-orbits.
($G \cdot x \simeq G$. i.e. $\mathrm{Stab}_G(x) = \{1\}$).

Ex The multigraded Hilbert scheme.

3. $G = (\mathbb{C}^*)^m \times \mu_{m_1} \times \cdots \times \mu_{m_n} \curvearrowright \mathbb{C}^n$.

$\rightsquigarrow$ $A := \mathbb{C}[x_1, \dots, x_n]$ is a $\Lambda := \chi(G) = \mathbb{Z}^{\oplus m} \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$ - graded ring.

weighted polynomial ring. character group of G.

$A = \bigoplus_{\lambda \in \Lambda} A_\lambda \supset I$: a homogeneous ideal. s.t. $\dim_{\mathbb{C}} (A/I)_\lambda = h(\lambda)$.

$h : \Lambda \to \mathbb{Z}_{\geq 0}$ a Hilbert function.

4. The invariant Hilbert scheme $\mathrm{Hilb}^G_h(X)$.

- Alexeev–Brion '05. G : a connected reductive algebraic group.
- Brion '10. G : any red. alg. group.

$\mathrm{Hilb}^G_h(X) \to X /\!\!/ G$. "the quotient-scheme map" an analogue of the Hilbert–Chow morphism.

§ Resolution of singularities of quotient varieties.

Definition. Y : an irreducible algebraic variety.

(1) $y \in Y$ is a singular point $\underset{\text{def}}{\Leftrightarrow}$ $\dim_{\mathbb{C}} T_y Y > \dim Y$.

(2) $f : \tilde{Y} \to Y$ is a resolution of singularities of Y. $\underset{\text{def}}{\Leftrightarrow}$ · $\tilde{Y}$ is smooth, · f is projective, birational.

· G : a reductive algebraic group $\curvearrowright$ X : an affine algebraic variety.

$\mathbb{C}[X] \supset \mathbb{C}[X]^G := \{ f \in \mathbb{C}[X] \mid g \cdot f = f,\ \forall g \in G \}$.

$\mathbb{C}[X] \curvearrowleft G$. "the ring of G-invariants" Action: $\forall x \in X$: $(g \cdot f)(x) := f(g^{-1}x)$.

$X \supset \pi^{-1}(y) \supset \exists!$ closed orbit.

$\pi \downarrow$ "the quotient morphism."

$X /\!\!/ G := \mathrm{Spec}(\mathbb{C}[X]^G) \ni y$.

Remark. (1) In general, $X /\!\!/ G$ is singular even if X is smooth.

(2) $X /\!\!/ G$ parametrizes closed G-orbits in X.

(3) If G is finite, $X/G = \{ G\text{-orbits in } X \}$.

Example. $G = \mu_n \curvearrowright \mathbb{C}^2$, $(t,s) \mapsto (\xi t, \xi^{-1}s)$.

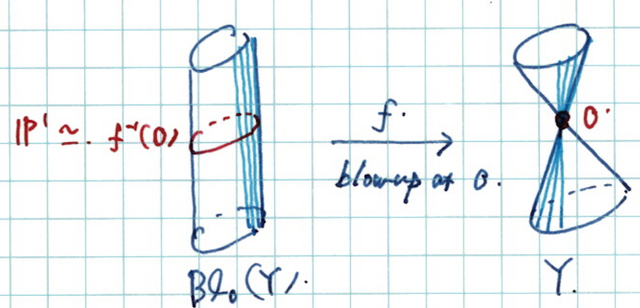
SL_2?

$$\mathbb{C}[\mathbb{C}^2]^G = \mathbb{C}[t,s]^{\mu_n} = \underbrace{\mathbb{C}[t^n, ts, s^n]}_{\substack{\mu_n\text{-invariants} \\ \text{with a relation.} \\ t^n \cdot s^n - (ts)^n = 0}} \simeq \mathbb{C}[x,y,z]/(xz - y^n).$$

$t^n \mapsto x$
$ts \mapsto y$
$s^n \mapsto z$

e.g. $(n=2)$.

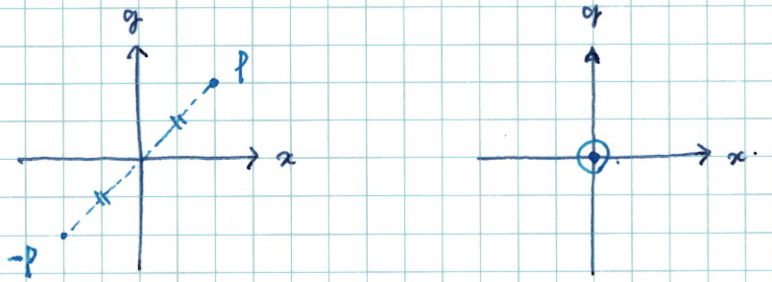
$$\mathbb{C}^2/\mu_2 \simeq \overset{Y}{\overset{\shortparallel}{(xz - y^2 = 0)}} \subset \mathbb{C}^3$$



$Bl_0(Y)$ as the invariant Hilbert scheme.

$\mathbb{C}^2$ "geometric quotient."
$\downarrow$
$\mathbb{C}^2/\mu_2$ = the space of μ_2-orbits in $\mathbb{C}^2$.

μ_2-orbits in $\mathbb{C}^2$:



$p \neq 0$. $\mathrm{Orb}_{\mu_2}(p) = \{p, -p\}$.

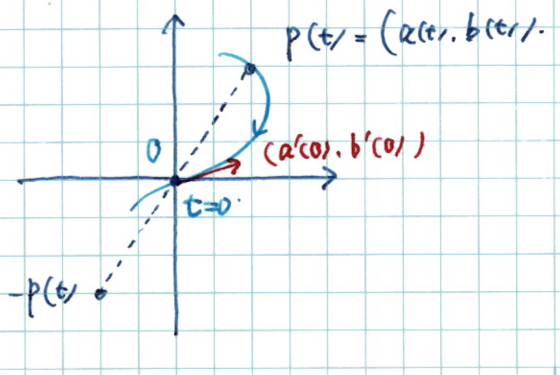
$p = 0$. $\mathrm{Orb}_{\mu_2}(0) = \{0\}$.

Observation.

$$\mathbb{C}^2 \supset \mathrm{Orb}_{\mu_2}(p(t)) \longleftrightarrow I_{\mathrm{Orb}_{\mu_2}(p(t))} \underset{\text{ideal.}}{\subset} \mathbb{C}[x,y].$$

$\lim_{t\to 0} \mathrm{Orb}_{\mu_2}(p(t)) = \{0\}$. singular point.
limit of a general point in the orbit space $\mathbb{C}^2/\mu_2$.

$\lim_{t\to 0} I_{\mathrm{Orb}_{\mu_2}(p(t))} = ?$.
limit of the defining ideal in the invariant Hilbert scheme $\mathrm{Hilb}^{\mu_2}_{h}(\mathbb{C}^2)$.
"flat limit"



$I_{\mathrm{Orb}(p(t))} = (b(t)x - a(t)y,\ x^2 - a(t)^2,\ xy - a(t)b(t),\ y^2 - b(t)^2)$.

$\cap$

$\mathbb{C}[x,y]$

$$\lim_{t\to 0} I_{\mathrm{Orb}(p(t))} = (I_{\mathrm{Orb}(p(t))} : (t^N))|_{t=0} \quad (\exists N \gg 1).$$

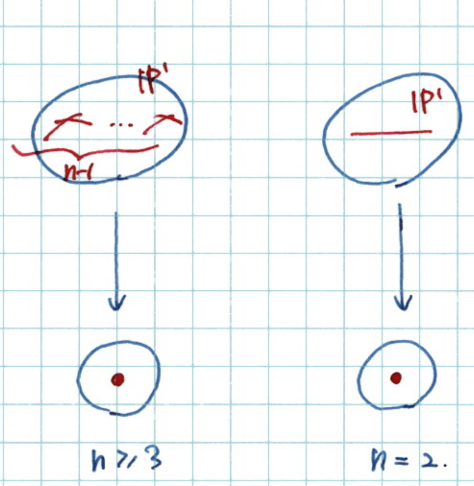
$$= \underbrace{(b'(0)x - a'(0)y)}_{\text{equation of the tangent line.}} + \underbrace{(x^2, xy, y^2)}_{\text{ideal of the orbit of the origin.}} =: J_{[a'(0):b'(0)]}$$

$$\mathrm{Hilb}^{\mu_2}_h(\mathbb{C}^2) = \{I_{\mathrm{Orb}(p(t))} \mid p(t)\neq 0\} \cup \{J_{[\alpha:\beta]} \mid [\alpha:\beta]\in\mathbb{P}^1\}.$$

The Hilbert-Chow morphism. $\gamma : \mathrm{Hilb}^{\mu_2}_h(\mathbb{C}^2) \longrightarrow \mathbb{C}^2/\mu_2$, $\begin{cases} I_{\mathrm{Orb}(p(t))} \mapsto \mathrm{Orb}(p(t)) \\ J_{[\alpha:\beta]} \mapsto \{0\}. \end{cases}$

$\mathbb{P}^1 \simeq \gamma^{-1}(0) \subset \mathrm{Hilb}^{\mu_2}_h(\mathbb{C}^2)$, $\quad \mathrm{Hilb}^{\mu_2}_h(\mathbb{C}^2) \simeq Bl_0(Y) \longrightarrow Y \simeq \mathbb{C}^2/\mu_2$.

γ is the minimal resolution.



__Extension.__ (1) $\mu_n \curvearrowright \mathbb{C}^2$ $(n \geq 3)$.

$J_k = I_k + (x,y)^k$.

$(\exists I_k \subset \mathbb{C}[x,y],\ 1 \leq k \leq n-1)$.

The calculation of $\mathrm{Hilb}^{\mu_n}_h(\mathbb{C}^2)$.
$=$ the calculation of the ideals I_k

(2) G: infinite $\curvearrowright X$.

- In general, $X /\!\!/ G$ is not an orbit space.
- G has infinitely many irreducible representations.

§. The invariant Hilbert scheme.

G: a reductive alg. group. V: a G-representation.

$\rightsquigarrow$ complete reducibility (Wyle). $\quad V \simeq \bigoplus_{M \in \mathrm{Irr}(G)} \mathrm{Hom}^G(M, V) \otimes_{\mathbb{C}} M$.

If $\dim_{\mathbb{C}} \mathrm{Hom}^G(M, V) < \infty$ for $\forall M \in \mathrm{Irr}(G)$,

$h_V : \mathrm{Irr}(G) \to \mathbb{Z}_{\geq 0}$, $M \mapsto h_V(M) := \dim_{\mathbb{C}} \mathrm{Hom}^G(M, V)$.

"the Hilbert function of V." $\quad$ the multiplicity of M in V.

Example. $V = \mathbb{C}[G]$ the regular representation.

$$\mathbb{C}[G] \simeq \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus \dim_{\mathbb{C}} M} \rightsquigarrow h_{\mathbb{C}[G]}(M) = \dim_{\mathbb{C}} M.$$

e.g. $G = \mathbb{C}^*$. $\mathrm{Irr}(G) \simeq \mathbb{Z}$.

Every irreducible representation of $\mathbb{C}^*$ is 1-dimensional. $\therefore h_{\mathbb{C}[G]} \equiv 1$

Definition. (The invariant Hilbert scheme)

G: a red. alg. group, X: an affine G-variety, h: a Hilbert function.

$$\mathrm{Hilb}^G_h(X) := \{ \underset{\substack{G\text{-invariant} \\ \text{cl. subscheme}}}{Z \subset X} \mid \underset{\substack{\circlearrowleft \\ G}}{\mathbb{C}[Z]} \simeq \bigoplus_{M \in \mathrm{Irr}(G)} M^{\oplus h(M)} \text{ as } G\text{-representations} \}.$$

Example. (1) $G = \{1\}$. $\rightsquigarrow h \equiv n$. ($\exists n \in \mathbb{Z}_{\geq 0}$).

$$\mathrm{Hilb}^G_h(X) = \{ \underset{\text{cl.}}{Z \subset X} \mid \mathbb{C}[Z] \simeq \mathbb{C}^{\oplus n} \} = \mathrm{Hilb}^n(X).$$

"the Hilbert scheme of n points on X"

(2) G: finite, $h = h_{\mathbb{C}[G]}$

$$\mathrm{Hilb}^G_{h_{\mathbb{C}[G]}}(X) = \{ \underset{G\text{-inv. cl.}}{Z \subset X} \mid \mathbb{C}[Z] \simeq \mathbb{C}[G] \} = G\text{-Hilb}(X).$$

"the G-Hilbert scheme"

Hilbert-Chow morphism.

$$\begin{array}{ccccc} & & X & & \\ & & \downarrow & & \\ \gamma: G\text{-Hilb}(X) & \longrightarrow & X/G & \overset{\text{open}}{\supset} & U = \{\text{free } G\text{-orbits}\}. \\ Z & \longmapsto & \mathrm{Supp}\, Z. & & \end{array}$$

γ is an isomorphism over U.

Thm. (1) (Ito-Nakamura). $\underset{\text{finite}}{G \subset SL_2}$, $X = \mathbb{C}^2 \Rightarrow \gamma$ is the minimal resolution.

(2) ($\underset{\text{abelian}}{\text{Nakamura}}$, Bridgeland-King-Reid). $\underset{\text{finite}}{G \subset SL_3}$, $X = \mathbb{C}^3 \Rightarrow \gamma$ is a crepant resolution

The quotient-scheme map. G: any red. alg. group. (not necessarily finite.)

$Z \in \mathrm{Hilb}^G_h(X) \rightsquigarrow \mathbb{C}[Z]^G \simeq \mathbb{C}^{\oplus h(0)}$. ($0 \in \mathrm{Irr}(G)$: the trivial rep.)

Put $n := h(0)$. $\quad \gamma: \mathrm{Hilb}^G_h(X) \to \mathrm{Hilb}^n(X /\!/ G)$ projective.

$$Z \mapsto Z /\!/ G.$$

(!) $n = 1. \Rightarrow \mathrm{Hilb}^n(X/\!/G) = X /\!/ G$.

Q. $\exists h$: a Hilbert function s.t. $h(0) = 1$ & γ is birational ?

Observation G: finite.

$$\begin{array}{ccccc} \mathrm{Hilb}^G_{h_{\mathbb{C}[G]}}(X) & & X & \supset \pi^{-1}(y) \simeq G & \rightsquigarrow h_{\pi^{-1}(y)} = \underline{h_{\mathbb{C}[G]}} \\ \| & & \pi \downarrow & & \\ G\text{-}\mathrm{Hilb}(X) & \xrightarrow{\gamma} & X/G & \supset U = \{\text{free } G\text{-orbits}\} \ni y & \\ \cup & & \cup & & \\ \gamma^{-1}(U) & \xrightarrow{\sim} & U & & \end{array}$$

G: arbitrary.

$$\begin{array}{ccccc} X & \supset & \pi^{-1}(U) & \supset \pi^{-1}(y) \circlearrowleft G & \rightsquigarrow h_{\pi^{-1}(y)} \\ \pi \downarrow & & \downarrow \text{ flat locus} & & \\ X/\!\!/G & \supset & U \ni y & & \end{array}$$

(!) Every fiber over U has the same Hilbert function. $=: h_X$.

$\pi^{-1}(y)/\!\!/G = \{y\}$.
$\rightsquigarrow h_X(0) = 1$.

Thm (Brion) G, h_X : as above.

$\gamma : \mathrm{Hilb}^G_{h_X}(X) \to X/\!\!/G$ is an isomorphism over U.

- $\mathcal{H}^m := \overline{\gamma^{-1}(U)} \subset \mathrm{Hilb}^G_{h_X}(X)$ with the reduced scheme structure.

"the main component" irreducible component

Q. Does $\gamma|_{\mathcal{H}^m}$ give a resolution of singularities of $X/\!\!/G$?

Thm (Terpereau). V: a finite-dimensional $\mathbb{C}$-vector space, $G = SL(V) \curvearrowright X = V^{\oplus m}$.

Put $\mathcal{H} = \mathrm{Hilb}^G_{h_X}(X)$.

(1) $\mathcal{H} \simeq \{0\}$. $\quad (m < \dim V)$

(2) $\mathcal{H} \simeq \mathbb{A}^1_{\mathbb{C}}$ $\quad (m = \dim V)$

(3) $\mathcal{H} \simeq Bl_0(\widetilde{Gr(\dim V, \mathbb{C}^m)})$. $\quad (m > \dim V)$.

In all cases, $\gamma : \mathcal{H} \to X/\!\!/G$ gives a resolution of singularities.

Remark. (1). The first fundamental theorem for SL, Weyl '39

$X = V^{\oplus m} \ni (v_1, \cdots, v_m)$, $n = \dim V$.

$m \geq n \Rightarrow$. $\mathbb{C}[X]^{SL(V)}$ is generated by

$\det(v_{i_1} | \cdots | v_{i_n})$, $1 \leq i_1, \cdots, i_n \leq m$.

(2). Other examples of (G, X) (with an infinite G) that the associated γ gives a resolution of singularities :

[Terpereau '14], [Becker '11], [Jansou-Ressayre '09], [Lehn-Terpereau '15], [K '20]

(3). $\exists$ Examples where γ is NOT a resolution.

[Terpereau '14] $\dim V > 2$, $G = GL(V)$, $SO(V)$, $O(V)$.

Remark. (1). The original motivation of Alexeev and Brion. (2005, G: connected).

was to study the moduli space of quasi-homogeneous G-varieties.

G-variety with a dense open G-orbit.

e.g. Moduli of spherical varieties.

normal G-variety with an open B-orbit, $B \subset G$ Borel.

(2). $\mathrm{Hilb}^G_h(X)$ (G: connected) is a generalization of the multigraded Hilbert scheme.

$$G: \text{connected} \curvearrowright X.$$
$$\cup$$
$$B = U \cdot T \quad (U: \text{unipotent part of } B,\ T: \text{maximal torus}).$$

Borel subgroup

- $\mathbb{C}[X]^U$ finitely generated. $\rightsquigarrow$ $X/\!/U$ is an affine T-variety. ($T \curvearrowright \mathbb{C}[X]^U$)

- $\Lambda^+ := \mathrm{Irr}(G) \subset X(B) = X(T) =: \Lambda$ (character group), $h: \Lambda^+ \to \mathbb{Z}_{\geq 0}$ given.

subset of dominant weights.

Define $\bar{h}: \Lambda \to \mathbb{Z}_{\geq 0}$ as.
$$\begin{cases} \bar{h}(\lambda) = h(\lambda) & \text{if } \lambda \in \Lambda^+ \\ \bar{h}(\lambda) = 0 & \text{if } \lambda \in \Lambda \setminus \Lambda^+ \end{cases}$$

Fact 1 $\mathrm{Hilb}^G_h(X)$ is a closed subscheme of $\mathrm{Hilb}^T_{\bar{h}}(X/\!/U)$.

Fact 2. Y: an affine G-variety. $X \subset Y$: a closed G-subscheme.
$\mathrm{Hilb}^G_h(X)$ is a closed subscheme of $\mathrm{Hilb}^G_h(Y)$

Take a T-equivariant embedding. $X/\!/U \hookrightarrow \mathbb{A}^N$.

$\rightsquigarrow$ $\mathrm{Hilb}^T_{\bar{h}}(X/\!/U) \subset \underline{\mathrm{Hilb}^T_{\bar{h}}(\mathbb{A}^N)}$.

multigraded Hilbert scheme.

§. The Cox realization.

Assume that an affine variety Y has two quotient presentations:

$$X/\!/G \simeq Y \simeq X'/\!/G'$$

$$\rightsquigarrow \quad \gamma: \mathrm{Hilb}^G_{h_X}(X) \longrightarrow X/\!/G$$
$$\hphantom{\rightsquigarrow \quad \gamma: \mathrm{Hilb}^G_{h_X}(X) \longrightarrow\ } \wr\!\!\mid$$
$$\gamma': \mathrm{Hilb}^{G'}_{h_{X'}}(X') \longrightarrow X'/\!/G'$$

In general, $\gamma \neq \gamma'$

Q. $\exists$ quotient presentation of Y s.t. the associated quotient-scheme map give a resolution of singularities of Y?

Definition (The Cox ring).

Y : a normal variety s.t. $\begin{cases} \mathbb{C}(Y)^* = \mathbb{C}^* \\ Cl(Y) \simeq \mathbb{Z}^{\oplus m} \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z} \end{cases}$

$$\mathrm{Cox}(Y) := \bigoplus_{[D] \in Cl(Y)} H^0(Y, \mathcal{O}_Y(D)).$$ "$Cl(Y)$-graded ring."

Example. $Y = \mathbb{P}^N$, H : a hyperplane, $Cl(Y) = \mathbb{Z}\cdot[H]$.

$$\mathrm{Cox}(Y) = \bigoplus_{m \in \mathbb{Z}} H^0(Y, mH), \quad \mathrm{Proj}(\mathrm{Cox}(Y)) = Y.$$

- By the grading, $G_Y := \mathrm{Hom}_{\mathbb{Z}}(Cl(Y), \mathbb{C}^*) \curvearrowright \mathrm{Cox}(Y)$

$$\simeq (\mathbb{C}^*)^m \times \mu_{m_1} \times \cdots \times \mu_{m_r}$$

$\mathrm{Cox}(Y)^{G_Y} = H^0(Y, \mathcal{O}_Y)$. "the degree 0 part"

$\therefore$ Y : affine, $\mathrm{Cox}(Y)$: finitely generated. $\Rightarrow$ $\mathrm{Spec}(\mathrm{Cox}(Y)) /\!/ G_Y \simeq Y$.

Y is "recovered" from its Cox ring.

- In this case, the quotient morphism

$$\pi_Y : \mathrm{Spec}(\mathrm{Cox}(Y)) \xrightarrow[/\!/ G_Y]{} Y$$

is called the <u>Cox realization</u> of Y.

Note: $y \in Y$ general, $\pi_Y^{-1}(y) \simeq G_Y$. $\therefore$ $h_X = h_{\mathbb{C}[G_Y]} \equiv 1$.

$(X = \mathrm{Spec}(\mathrm{Cox}(Y)))$.

The invariant Hilbert scheme associated to the Cox realization.

$$\gamma_Y : \mathcal{H}_Y := \mathrm{Hilb}^{G_Y}_{h_{\mathbb{C}[G_Y]}}(\mathrm{Spec}(\mathrm{Cox}(Y))) \to Y$$

<u>Q.</u> Does γ_Y give a resolution of singularities of Y?

Example. $Y = (xz - y^n = 0) \subset \mathbb{C}^3$.

- $Cl(Y) \simeq \mathbb{Z}/n\mathbb{Z} \rightsquigarrow G_Y \simeq \mu_n$.
- $R := \mathrm{Cox}(Y) \simeq \mathbb{C}[t, s]$ polynomial ring

$$R = \bigoplus_{d \in \mathbb{Z}/n\mathbb{Z}} R_d. \quad R_0 = \mathbb{C}[Y], \quad R_d = R_0 \otimes_{\mathbb{C}} \langle t^d, s^{n-d} \rangle_{\mathbb{C}} \ (d > 0)$$

- Cox realization: $\mathrm{Spec}(\mathrm{Cox}(Y)) / G_Y \simeq \mathbb{C}^2/\mu_n$.
- $\mathcal{H}_Y = \mu_n\text{-}\mathrm{Hilb}(\mathbb{C}^2) \xrightarrow[\gamma_Y]{} \mathbb{C}^2/\mu_n$. minimal resolution by [Ito-Nakamura]

§ Nilpotent orbits in $\mathfrak{sl}_n$.

$A \in \mathfrak{sl}_n$ nilpotent. $\quad A \underset{\text{conjugate}}{\sim} \begin{pmatrix} J_{d_1} & & \\ & \ddots & \\ & & J_{d_n} \end{pmatrix} =: A_{\mathbf{d}}$, $\quad J_{d_i}$: Jordan matrix of size d_i.

$\rightsquigarrow$ $\mathbf{d} = [d_1, \cdots, d_n]$ is a partition of n. $\quad d_1 \geq \cdots \geq d_n \geq 1$, $\sum d_i = n$.

$$\{\text{Nilpotent orbits in } \mathfrak{sl}_n\} \overset{1:1}{\leftrightarrow} \{\text{Partitions of } n\}.$$

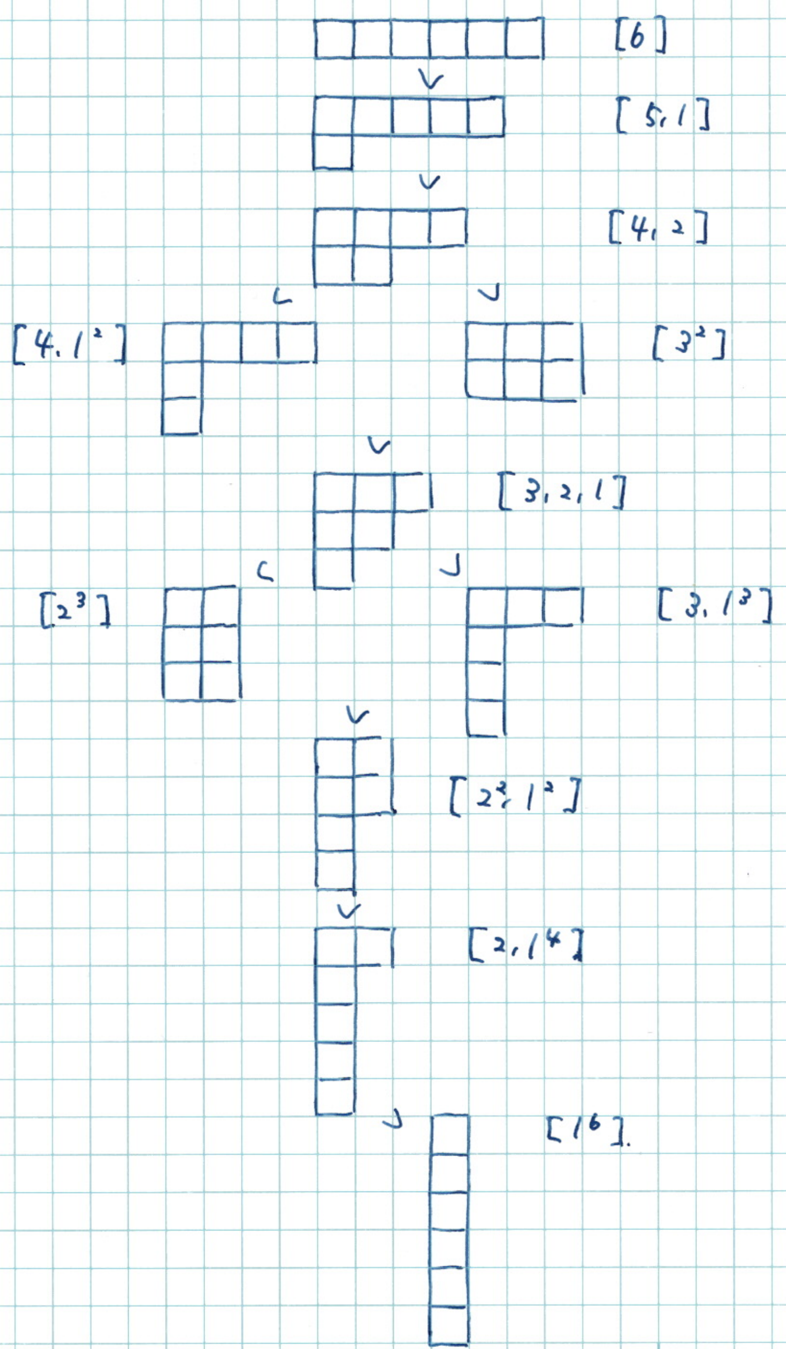
$$\mathcal{O}_{\mathbf{d}} := \underset{\text{conjugate action}}{SL_n \cdot A_{\mathbf{d}}} \quad \mapsfrom \quad \mathbf{d}$$

$$\overline{\mathcal{O}}_{\mathbf{d}} \subset \mathfrak{sl}_n. \quad \mathrm{Sing}(\overline{\mathcal{O}}_{\mathbf{d}}) = \overline{\mathcal{O}}_{\mathbf{d}} \setminus \mathcal{O}_{\mathbf{d}} = \bigcup_{\mathbf{d}' < \mathbf{d}} \mathcal{O}_{\mathbf{d}'}$$

Order relation on the set of partitions.

$\mathbf{d} = [d_1, \cdots, d_n]$, $\mathbf{d}' = [d_1', \cdots, d_l']$. $\quad \mathbf{d}' \leq \mathbf{d} \underset{\text{def}}{\Leftrightarrow} \sum_{i=1}^{j} d_i' \leq \sum_{i=1}^{j} d_i$, $\forall j$.

Example. $(n=6)$.



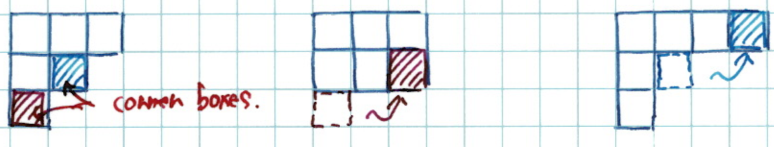
Remark. (1) (Gerstenhaber). $\lambda' \leq \lambda \Leftrightarrow \mathcal{O}_{\lambda'} \subseteq \overline{\mathcal{O}}_{\lambda}$

(2). $\overline{\mathcal{O}}_{[n]} = \bigcup_{\lambda:\ \text{partition of } n} \mathcal{O}_{\lambda}$ "the nilpotent cone"

(3) (Kraft–Procesi) $\lambda' < \lambda$ and $\nexists\, \lambda''$ s.t. $\lambda' < \lambda'' < \lambda$

$\Leftrightarrow$ The Young diagram of λ is obtained by the one of λ' by raising a corner box to the first allowable position.

e.g. $\lambda' = [3,2,1]$



(4) $\dim \mathcal{O}_{\lambda} = n^2 - \sum_{i=1}^{l} p_i^2$, $\mathbb{P} = [p_1, \dots, p_l]$ dual partition.

$(p_i = \#\{ j \mid \lambda_j \geq i \})$.

e.g. $\lambda = [3,1] \overset{}{\underset{\text{dual}}{\leftrightarrow}} \mathbb{P} = [2, 1^2]$ $(n=4)$.

$\dim \mathcal{O}_{[3,1]} = 16 - (4+1+1) = 10$.

$\dim \mathcal{O}_{[2,1^2]} = 16 - (9+1) = 6$.

The Springer resolution.

$P \subset SL_n$ parabolic subgroup. $\leftrightarrow$ SL_n/P flag variety.

P is a polarization of $\mathcal{O}_{\lambda}$. $\overset{}{\underset{\text{def.}}{\Leftrightarrow}}$ $\mathfrak{n}(\mathfrak{p}) \cap \mathcal{O}_{\lambda} \underset{\text{dense open}}{\subset} \mathfrak{n}(\mathfrak{p})$,

$\mathfrak{n}(\mathfrak{p})$: the nilradical of $\mathfrak{p} = \mathrm{Lie}(P)$.

- $SL_n \times^{P} \mathfrak{n}(\mathfrak{p}) := (SL_n \times \mathfrak{n}(\mathfrak{p}))/P$.

  Action: $g \cdot (X, x) := (Xg^{-1}, gxg^{-1})$.

- $T^*(SL_n/P) \simeq SL_n \times^{P} \mathfrak{n}(\mathfrak{p})$, $(X, XxX^{-1}) \mapsfrom (X, x)$.

  $\shortparallel$

  $\{ (\{V_i\}, x) \in (SL_n/P) \times \overline{\mathcal{O}}_{\lambda} \mid xV_i \subset V_{i-1} \}$.

- $$\begin{array}{ccc} (SL_n/P) \times \overline{\mathcal{O}}_{\lambda} & \xrightarrow{pr_2} & \\ \cup & & \overline{\mathcal{O}}_{\lambda} \overset{?}{\supset} \mathcal{O}_{\lambda} \\ T^*(SL_n/P) & \xrightarrow[\eta]{} & \end{array}$$ projective, birational.

  "the Springer resolution"

Example. $\mathbf{d} = [2,1]$, $\mathcal{O}_{[2,1]}$ has 3 polarizations.

① $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}$ $0 \subset V_1 = \langle e_1, e_4 \rangle \subset V_2 = V_1 \oplus \langle e_2 \rangle \subset V_3 = V_2 \oplus \langle e_3 \rangle = \mathbb{C}^4$

flag of type $(2,1,1)$. $\circlearrowleft \mathfrak{S}_3$.

$$P_1 = \begin{bmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ * & * & * & * \end{bmatrix}$$

② $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline & 4 & \\ \hline \end{array}$ $0 \subset V_1 = \langle e_1 \rangle \subset V_2 = V_1 \oplus \langle e_2, e_4 \rangle \subset \mathbb{C}^4$.

flag of type $(1,2,1)$

$$P_2 = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{bmatrix}$$

③ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline & & 4 \\ \hline \end{array}$ $0 \subset V_1 = \langle e_1 \rangle \subset V_2 = V_1 \oplus \langle e_2 \rangle \subset \mathbb{C}^4$

flag of type $(1,1,2)$

$$P_3 = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

§ Cox realization of $\overline{\mathcal{O}}_{\mathbf{d}} \subset \mathfrak{sl}_n$ and the associated invariant Hilbert scheme.

· (Kraft–Procesi) $\operatorname{codim}(\overline{\mathcal{O}}_{\mathbf{d}} \setminus \mathcal{O}_{\mathbf{d}}) \geq 2 \Rightarrow \operatorname{Cox}(\overline{\mathcal{O}}_{\mathbf{d}}) \simeq \operatorname{Cox}(\underbrace{\mathcal{O}_{\mathbf{d}}}_{\simeq SL_n/\operatorname{Stab}(A_{\mathbf{d}})})$.

· (Fu) $c := \gcd(d_1, \dots, d_n)$, $k := \#\{\text{distinct } d_i\}$.

$$\Rightarrow \operatorname{Cl}(\mathcal{O}_{\mathbf{d}}) \simeq \mathbb{Z}^{\oplus k-1} \oplus \mathbb{Z}/c\mathbb{Z}.$$

By [ADHL], $F := \operatorname{Stab}_{SL_n}(A_{\mathbf{d}})$, $F_1 := \bigcap_{\chi \in \mathcal{X}(F)} \operatorname{Ker}(\chi) \subset F$.

$\Rightarrow \operatorname{Cl}(\mathcal{O}_{\mathbf{d}}) \simeq \mathcal{X}(F)$,

$$\operatorname{Cox}(\underbrace{\mathcal{O}_{\mathbf{d}}}_{\simeq SL_n/F}) \simeq \mathbb{C}[SL_n]^{F_1} = \bigoplus_{\chi \in \mathcal{X}(F)} \mathbb{C}[SL_n]^{F_1}_{\chi}.$$

$$\cup$$

$$\mathbb{C}[SL_n]^{F_1}_0 = \mathbb{C}[SL_n]^F \simeq \mathbb{C}[\overline{\mathcal{O}}_{\mathbf{d}}].$$

Thm (K., K-Nagai).

In the following cases, $\gamma_Y|_{\mathcal{H}^{m}_{Y}}$ gives a resolution of singularities:

$Y = \overline{\mathcal{O}}_{\mathbf{d}} \subset \mathfrak{sl}_n$ with partition

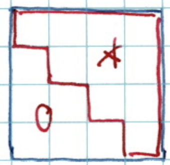
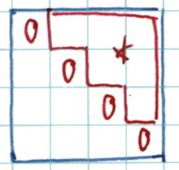
(A) $\mathbf{d} = [a^k]$ $(a \geq 2, k \geq 1)$.

(B) $\mathbf{d} = [2^k, 1^{n-2k}]$ $(n-2k > 0)$.

Outline. $\alpha = [n]$ (special case of (A1)).

Young tableau: $\boxed{1 \mid \cdots \mid n}$ $\quad 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \mathbb{C}^n =: V$

full flag variety.

$P = B =$  , $\mathfrak{sl}_n \supset \mathfrak{w} =$ 

$\eta : SL_n \times^B \mathfrak{w} \longrightarrow \bar{\mathcal{O}}_{[n]}$ the Springer resolution.

① Cox ring of $\bar{\mathcal{O}}_{[n]}$:

$Cl(\bar{\mathcal{O}}_{[n]}) \simeq \mathbb{Z}/n\mathbb{Z}$.

$$R := Cox(\bar{\mathcal{O}}_{[n]}) \simeq \mathbb{C}[SL_n]^{F_1} = \bigoplus_{\gamma=0}^{n-1} \mathbb{C}[SL_n]^{F_1}_{\gamma}, \quad R_\gamma = \mathbb{C}[SL_n]^{F_1}_{\gamma}.$$

$\circlearrowleft \mu_n$.

$$\cdot\ F = Stab_{SL_n}(A_{[n]}) = \left\{ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & x_{11} & \ddots & \vdots \\ & & \ddots & x_{12} \\ 0 & & & x_{11} \end{pmatrix} : \begin{array}{l} x_{11}^n = 1, \\ x_{i,j} = x_{i+1,j+1} \ (i \leq j) \end{array} \right\}.$$

$\cup$

$F_1 = \{ X \in F \mid x_{11} = 1 \}$.

$$R_\gamma = R_0 \otimes_{\mathbb{C}} \langle \det(X^{(i_1, \dots, i_\gamma)}_{(1, \dots, \gamma)}) \mid 1 \leq i_1 < \cdots < i_\gamma \leq n \rangle_{\mathbb{C}} \simeq R_0 \otimes_{\mathbb{C}} \wedge^\gamma V.$$

(Graham). $\quad (V = \mathbb{C}^n$, std. rep.$)$

② Invariant Hilbert scheme of the Cox realization.

$$\begin{tikzcd}
\mathcal{H} = Hilb^{\mu_n}_{h}(\operatorname{Spec} R) \arrow[rd, "\gamma"] & \\
\exists!\, Z_{[n]} \in \mathcal{H}^m = \overline{\gamma^{-1}(\mathcal{O}_{[n]})} \arrow[u, phantom, "\cup"] \arrow[r, "\gamma|_{\mathcal{H}^m}"] & \bar{\mathcal{O}}_{[n]} \arrow[d, equal] \\
SL_n \times^B \mathfrak{w} \arrow[u, "\exists\, \wr"] \arrow[r, "\eta"] \arrow[ur, phantom, "\circlearrowright"] & \bar{\mathcal{O}}_{[n]}
\end{tikzcd}$$

$\bar{\mathcal{O}}_{[n]} \supset \mathcal{O}_{[n]} = SL_n \cdot A_{[n]} \ni A_{[n]} = \gamma(Z_{[n]})$.

$SL_n \times^B \mathfrak{w} \subset (SL_n/B) \times \bar{\mathcal{O}}_{[n]}$.

· The classifying map. $\varphi : \mathcal{H} \to \mathbb{P} := \prod_{\gamma=1}^{n-1} \mathbb{P}(\wedge^\gamma V)$ $\quad SL_n$-equiv.

$\mathcal{H} \ni Z$.

$$R/I_Z = \bigoplus_{\gamma=0}^{n-1} \underbrace{R_\gamma / I_{Z,(\gamma)}}_{\simeq\, \mathbb{C}} \quad (h \equiv 1).$$

$$\rightsquigarrow \boxed{\begin{tikzcd} R_\gamma \arrow[r, two heads] & R_\gamma / I_{Z,(\gamma)} \simeq \mathbb{C} \\ \wedge^\gamma V \arrow[u, phantom, "\cup"] \arrow[ru, two heads] & \end{tikzcd}}$$

defines a point in $\mathbb{P}(\wedge^\gamma V)$.

$$\varphi(Z_{[n]}) = (\underbrace{[e_1], [e_1\wedge e_2], \dots, [e_1\wedge\cdots\wedge e_{n-1}]}_{\text{highest weight vectors}}).$$

$\rightsquigarrow \mathrm{Stab}_{SL_n}(\varphi(Z_{[n]})) = B.$

$\rightsquigarrow \varphi(\mathcal{H}^m) \simeq SL_n/B \subset \mathbb{P}.$

$\varphi|_{\mathcal{H}^m}$, $\mathcal{H}^m$, $N := \varphi^{-1}(\mathbb{1})$, $\mathbb{1}$ class of the identity matrix.

bijective onto the image.

$$\begin{array}{ccccc}
SL_n\times^B N & \longrightarrow & \mathcal{H}^m & \xrightarrow{\varphi\times r} & (SL_n/B)\times \bar{Q}_{[n]} \\
{\scriptstyle \mathrm{id}\times r}\downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle pr_2} \\
SL_n\times^B W & & \xrightarrow{\eta} & & \bar{Q}_{[n]}
\end{array}$$

Zariski's Main Theorem. $\Rightarrow$ $\mathcal{H}^m \simeq SL_n\times^B W$.

$\square$