Kneser彩色関数とツリーの完全不変量

可換代数と組合せ論セミナー

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1. Paper: Universal graph series, chromatic functions, and their index theory
   arXiv: 2403.09985
   Author: Tsuyoshi Miezaki, Akihiro Munemasa, Yusaku Nishimura, Tadashi Sakuma, and Shuhei Tsujie

2. Paper: The Kneser chromatic function distinguishes trees
   arXiv: 2409.20478
   Author: Yusaku Nishimura

Proper coloring

- $G = (V(G), E(G))$: simple graphs

Definition

A function $f : V(G) \longrightarrow \mathbb{N}$ is called a **proper coloring** if $f(u) \neq f(v)$ for all $\{u, v\} \in E(G)$.

Definition (Graph homomorphism)

$f : V(G) \to V(H)$ is a homomorphism if for any $\{x, y\} \in E(G)$, $\{f(x), f(y)\} \in E(H)$.

Proper coloring of G

$\iff$

Graph homomorphism from G to the complete graph

Chromatic polynomial

- $\mathrm{Hom}(G, H)$: the set of graph homomorphisms from G to H.
- K_n: the complete graph with n vertices.

Definition

The chromatic polynomial of G is

$$\chi(G, n) = \sharp\mathrm{Hom}(G, K_n).$$

Example

Let T be a tree with m vertices. Then $\chi(T, n) = n(n-1)^{m-1}$.

Chromatic symmetric function

- $K_{\mathbb{N}}$: the complete graph whose vertex set is the set of natural numbers.

Definition (Stanley (1995))

The chromatic symmetric function of G is

$$X_{K_{\mathbb{N}}}(G) := X_{K_{\mathbb{N}}}(G, x) := \sum_{\varphi \in \mathrm{Hom}(G, K_{\mathbb{N}})} \prod_{v \in V_G} x_{\varphi(v)}.$$

Remark

$\mathbf{1}^n := (\underbrace{1, \cdots, 1}_{n}, 0, \cdots)$. Then,

$$X_{K_{\mathbb{N}}}(G, \mathbf{1}^n) = \sum_{\varphi \in \mathrm{Hom}(G, K_n)} 1 = \sharp \mathrm{Hom}(G, K_n) = \chi(G, n).$$

Example of chromatic symmetric function

Example (Stanley (1995))

$$X_{K_{\mathbb{N}}}(G_1) = 4x_1x_2^2x_3^2 + 4x_1^2x_2^2x_3 + 4x_1^2x_2x_3^2 + 24x_1x_2x_3x_4^2 + \cdots$$

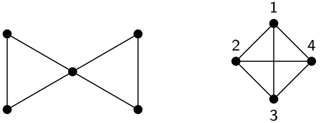


Figure: Graphs G_1 and K_4

$$\chi(G_1, 3) = X_{K_{\mathbb{N}}}(G_1, \mathbf{1}^3) = 4 + 4 + 4 = 12$$

Chromatic symmetric function

Theorem (Power sum expansion)

- $G = (V(G), E(G))$: a simple graph
- $|V(G)| = n$.
- S: any subset of $E(G)$.

Define $\lambda(S)$ the partition of n whose parts are equal to the vertex sizes of the connected components of the spanning subgraph of G with edge set S. Then,

$$X_{K_{\mathbb{N}}}(G) = \sum_{S \subset E} (-1)^{|S|} p_{\lambda(S)}$$

Conjecture (Stanley (1995))

If T_1 and T_2 are non-isomorphic trees then

$$X_{K_{\mathbb{N}}}(T_1) \neq X_{K_{\mathbb{N}}}(T_2).$$

Chromatic symmetric function

However, $X_{K_{\mathbb{N}}}(\bullet)$ is not a complete invariant for graphs.

Example (Stanley (1995))

G_1 and G_2 are non-isomorphic but

$$X_{K_{\mathbb{N}}}(G_1) = X_{K_{\mathbb{N}}}(G_2)$$

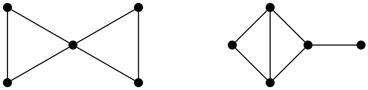


Figure: Graphs G_1 and G_2

k-fold proper coloring

- $G = (V(G), E(G))$: a simple graph
- $\binom{\mathbb{N}}{k} := \{I \subset \mathbb{N} : |I| = k\}$

Definition

A function $f : V(G) \longrightarrow \binom{\mathbb{N}}{k}$ is called a k-**fold proper coloring** if $f(u) \cap f(v) = \emptyset$ for every pair of adjacent vertices.

Example

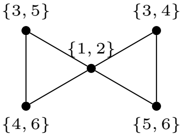


Figure: 2-fold proper coloring

k-fold proper coloring

$\Longleftrightarrow$ graph homomorphism from G to the Kneser graph

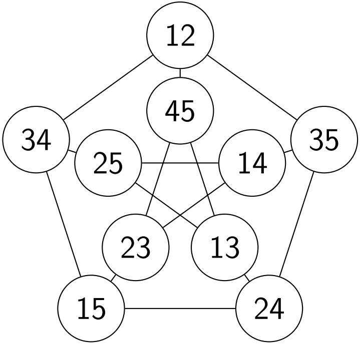
Kneser graph

Definition

The Kneser graph $K_{\mathbb{N},k}$ is the graph whose vertex set is $\binom{\mathbb{N}}{k}$ and for any $A, B \in \binom{\mathbb{N}}{k}$, $A \sim B$ if and only if $A \cap B = \emptyset$.

Remark

$K_{\mathbb{N},1}$ is the complete graph whose vertex set is the set of natural numbers.



Kneser chromatic function

Definition

- $K_{\mathbb{N},k}$: the Kneser graph
- x_u $(u \in \binom{\mathbb{N}}{k})$: indeterminates

For any graph G, **the Kneser-chromatic function** $X_{K_{\mathbb{N},k}}(G)$ is defined as follows:

$$X_{K_{\mathbb{N},k}}(G) := X_{K_{\mathbb{N},k}}(G,x) := \sum_{\varphi \in \mathrm{Hom}(G,K_{\mathbb{N},k})} \prod_{v \in V_G} x_{\varphi(v)}.$$

- $X_{K_{\mathbb{N},1}}(\bullet) = X_{K_{\mathbb{N}}}(\bullet)$ is the chromatic symmetric function.
- $X_{K_{\mathbb{N},k+1}}(G)$ is a stronger invariant than $X_{K_{\mathbb{N},k}}(\bullet)$:
  If $X_{K_{\mathbb{N},k+1}}(G) = X_{K_{\mathbb{N},k+1}}(H)$, then $X_{K_{\mathbb{N},k}}(G) = X_{K_{\mathbb{N},k}}(H)$.

Example of Kneser chromatic function

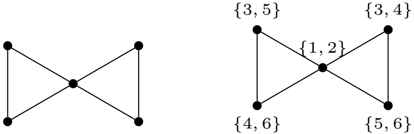


Figure: Graphs G_1 and $H' \hookrightarrow K_{\mathbb{N},2}$

$$
\begin{aligned}
X_{K_{\mathbb{N},2}}(G_1) &= \sum_{\varphi \in \mathrm{Hom}(G_1, K_{\mathbb{N},2})} \prod_{v \in V(G_1)} x_{\varphi(v)} \\
&= \cdots + 4x_{\{1,2\}}x^2_{\{3,4\}}x^2_{\{5,6\}} \\
&+ 24x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x^2_{\{7,8\}} + 8x^2_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,7\}} + 8x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}}x_{\{4,6\}} \\
&+ 48x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,9\}}x_{\{7,8\}} + 8x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,7\}}x_{\{4,6\}} + 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}} \\
&+ 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,8\}}x_{\{3,7\}} + 120x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{7,8\}}x_{\{9,10\}} + \cdots
\end{aligned}
$$

By $8x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}}x_{\{4,6\}}$, there are 8 homomorphisms from G_1 to H'.

Example of Kneser chromatic function

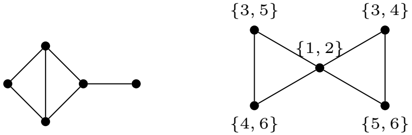


Figure: Graphs G_2 and $H' \hookrightarrow K_{\mathbb{N},2}$

$$\begin{aligned} X_{K_{\mathbb{N},2}}(G_2) = \cdots &+ 4x_{\{1,2\}}x_{\{3,4\}}^2x_{\{5,6\}}^2 + 24x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{7,8\}}^2 + x_{\{1,2\}}^2x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}} \\ &+ 4x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,5\}}x_{\{3,7\}} + 48x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,9\}}x_{\{7,8\}} \\ &+ 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{5,7\}}x_{\{7,8\}} + 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,8\}}x_{\{3,7\}} \\ &+ 0x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}}x_{\{4,6\}} + \cdots \end{aligned}$$

By $0x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}}x_{\{4,6\}}$, there are no homomorphisms from G_2 to H. Therefore, $X_{K_{\mathbb{N},2}}(G_1) \neq X_{K_{\mathbb{N},2}}(G_2)$

The generalization of Stanley's conjecture

Theorem (Miezaki et al. (2025+))

$\{X_{K_{\mathbb{N},k}}(\bullet)\}_{k\in\mathbb{N}}$ is a complete invariant for finite graphs.

Question (Miezaki et al. (2025+))

Is there a integer k such that $X_{K_{\mathbb{N},k}}(\bullet)$ is a complete invariant for trees?

Stanley's conjecture states that $X_{K_{\mathbb{N},k}}(\bullet)$ is a complete invariant for trees when $k = 1$.

Theorem (N. (2025+))

$X_{K_{\mathbb{N},2}}(\bullet)$ is a complete invariant for trees.

The property of Kneser chromatic function

The property of Kneser chromatic function

Theorem (Miezaki et al. (2025+))

Let $\{K_{\mathbb{N},k}\}_{k\in\mathbb{N}}$ be a Kneser graph series. Then

$$\{X_{K_{\mathbb{N},k}}(\bullet)\}_{k\in\mathbb{N}}$$

is a complete invariant for finite graphs.

Theorem (Power sum expansion, Miezaki et al. (2025+))

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S\subset E(G)} (-1)^{|S|} \sum_{\lambda\in\mathcal{A}^{(k)}_{G_S}} p_\lambda$$

Universal graph series

Definition

- $N \subset \mathbb{N}$
- $\{H_n\}_{n \in N}$: a family of graphs.

We say $\{H_n\}_{n \in N}$ is **a universal graph series** if for any simple graph G there exists n such that G is an induced subgraph of H_n.

$\{K_{\mathbb{N},k}\}_{k=1}^{\infty}$ is universal graph series([Hamburger–Por–Walsh (2009)]).

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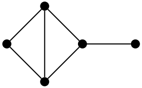

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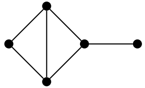

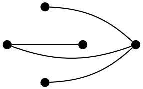

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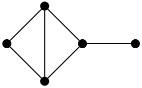

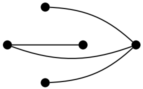

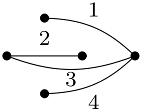

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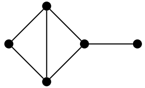

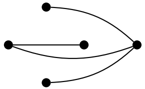

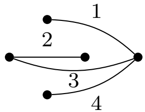

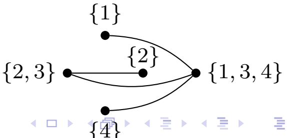

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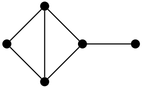
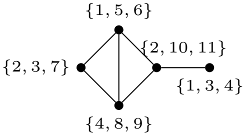
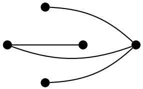
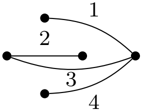
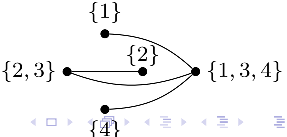










H-chromatic function

Definition

- $H = (V(H), E(H))$: simple graph
- x_u $(u \in V(H))$: indeterminates.

For any graph G, **the H-chromatic function** $X_H(G)$ is defined as follows:

$$X_H(G) := X_H(G, x) := \sum_{\varphi \in \mathrm{Hom}(G,H)} \prod_{v \in V_G} x_{\varphi(v)}.$$

Definition

- $H = \{H_n\}_{n \in N}$: a universal graph series

We define **the universal H-chromatic function** as follows:

$$\{X_{H_n}(G)\}_{n \in N}$$

The property of universal chromatic function

Theorem (Miezaki et al. (2025+))

Let $H = \{H_n\}_{n \in N}$ be a universal graph series. Then

$$\{X_{H_n}(\bullet)\}_{n \in N}$$

is a complete invariant for finite graphs.

Example

- $\{X_{K_{\mathbb{N},k}}(\bullet)\}_{k \in \mathbb{N}}$ is a complete invariant for finite graphs.
  - $X_{K_{\mathbb{N},1}}(\bullet) = X_{K_{\mathbb{N}}}(\bullet)$ is the chromatic symmetric function.
  - $X_{K_{\mathbb{N},k+1}}(\bullet)$ is a stronger invariant than $X_{K_{\mathbb{N},k}}(\bullet)$.

Universal chromatic function

Theorem (Miezaki et al. (2025+))

Let $H = \{H_n\}_{n \in N}$ be a universal graph series. Then

$$\{X_{H_n}(\bullet)\}_{n \in N}$$

is a complete invariant for finite graphs.

Lemma (Godsil and Royle (2001) p.128 Excercise 11)

Let G_1 and G_2 be finite graphs. If $|\mathrm{Hom}(G_1, F)| = |\mathrm{Hom}(G_2, F)|$ for any finite graph F, then G_1 and G_2 are isomorphic.

Universal chromatic function

Theorem (Miezaki et al. (2025+))

Let $H = \{H_n\}_{n \in N}$ be a universal graph series. Then

$$\{X_{H_n}(\bullet)\}_{n \in N}$$

is a complete invariant for finite graphs.

Proof of Theorem.

Let G_1 and G_2 be finite graphs and assume that $X_{H_n}(G_1) = X_{H_n}(G_2)$ for all $n \in N$. Suppose F is any graph. Since $\{H_n\}_{n \in N}$ is a universal graph series, there exists $n_f \in N$ such that F is an induced subgraph of H_{n_f}. Then, $|\mathrm{Hom}(G_1, F)|$ and $|\mathrm{Hom}(G_2, F)|$ are determined by $X_{H_{n_f}}(G_1)$ and $X_{H_{n_f}}(G_2)$. Hence, we obtain $|\mathrm{Hom}(G_1, F)| = |\mathrm{Hom}(G_2, F)|$. $\square$

The property of Kneser chromatic function

Power sum expansion of $X_{K_{\mathbb{N},k}}(G,x)$

Theorem

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}^{(k)}_{G_S}} p_\lambda$$

Example

$$X_{K_{\mathbb{N},1}}(\;) = p_3 - 2p_{2,1} + p_{1,1,1}$$

$$X_{K_{\mathbb{N},2}}(\;) = \sum_{\lambda \in \mathcal{A}^{(2)}_{3K_1}} p_\lambda - 2 \sum_{\lambda \in \mathcal{A}^{(2)}_{K_2 \sqcup K_1}} p_\lambda + \sum_{\lambda \in \mathcal{A}^{(2)}_{P_3}} p_\lambda$$

$$= p - 2p - 2p + p + p + p + p \,.$$

Weak homomorphism

Definition

Define

$$\mathrm{Hom}^{\mathrm{w}}(G,H) := \left\{ \varphi\colon V(G) \to V(H) \;\middle|\; \begin{array}{c} \text{If } \{u,v\} \in E(G) \text{ then} \\ \{\varphi(u),\varphi(v)\} \in E(H) \\ \text{or } \varphi(u) = \varphi(v) \end{array} \right\}$$

A map in $\mathrm{Hom}^{\mathrm{w}}(G,H)$ is called a **weak homomorphism**.

Definition

Define

$$W_H(G) := W_H(G,x) := \sum_{\varphi \in \mathrm{Hom}^{\mathrm{w}}(G,H)} \prod_{v \in V(G)} x_{\varphi(v)}.$$

Weak complement expansion

Proposition (Miezaki et al. (2025+))

$$X_H(G) = \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{H}}(G_S),$$

where G_S denotes the spanning subgraph of G with edge set S and $\overline{H}$ denotes the complement of H.

Proof.

$$\begin{aligned}\sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{H}}(G_S) &= \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\varphi \in \mathrm{Hom}^{\mathrm{w}}(G_S, \overline{H})} \prod_{v \in V(G)} x_{\varphi(v)} \\ &= \sum_{\varphi:\ V(G) \to V(H)} \sum_{S \subset E_\varphi} (-1)^{|S|} \prod_{v \in V(G)} x_{\varphi(v)},\end{aligned}$$

where

$$E_\varphi = \{\, \{u, v\} \in E(G) \mid \{\varphi(u), \varphi(v)\} \notin E(H) \,\}.$$

Weak complement expansion

Proof.

Since

$$\sum_{S \subset E_\varphi} (-1)^{|S|} = \begin{cases} 1 & \text{if } E_\varphi = \emptyset. \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{\varphi\colon V(G) \to V(H)} \sum_{S \subset E_\varphi} (-1)^{|S|} \prod_{v \in V(G)} x_{\varphi(v)} = X_H(G).$$

Therefore,

$$X_H(G) = \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{H}}(G_S)$$

□

Weak complement expansion of $X_{K_{\mathbb{N},k}}$

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{K_{\mathbb{N},k}}}(G_S)$$

Especially, when $k = 1$,

$$\begin{aligned}
X_{K_{\mathbb{N},1}}(G) &= \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{K_{\mathbb{N},1}}}(G_S) \\
&= \sum_{S \subset E} (-1)^{|S|} \prod_{i=1}^{m_S} W_{\overline{K_{\mathbb{N},1}}}(G_i) \\
&= \sum_{S \subset E} (-1)^{|S|} \prod_{i=1}^{m_S} \sum_{i \in \mathbb{N}} x_i^{|G_i|} \\
&= \sum_{S \subset E} (-1)^{|S|} p_{\lambda(S)},
\end{aligned}$$

where $G_S = G_1 \sqcup G_2 \cdots \sqcup G_{m_S}$.

Example of Kneser chromatic function

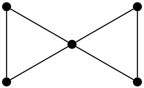


Figure: Graph G_1

$$
\begin{aligned}
X_{K_{\mathbb{N},2}}(G_1) = \cdots &+ 4x_{\{1,2\}}x^2_{\{3,4\}}x^2_{\{5,6\}} + 4x_{\{1,2\}}x^2_{\{3,5\}}x^2_{\{4,6\}} \\
&+ 24x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x^2_{\{7,8\}} + 8x^2_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,7\}} + 8x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,5\}}x_{\{4,6\}} \\
&+ 48x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,9\}}x_{\{7,8\}} + 8x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{3,7\}}x_{\{4,6\}} + 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}} \\
&+ 16x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{1,8\}}x_{\{3,7\}} + 120x_{\{1,2\}}x_{\{3,4\}}x_{\{5,6\}}x_{\{7,8\}}x_{\{9,10\}} + \cdots
\end{aligned}
$$

The coefficient of $x_{\{1,2\}}x^2_{\{3,4\}}x^2_{\{5,6\}}$ and $x_{\{1,2\}}x^2_{\{3,5\}}x^2_{\{4,6\}}$ must be the same, because

$$\{\sigma(\{1,2\}), \sigma(\{3,4\}), \sigma(\{5,6\})\} = \{\{1,2\}, \{3,5\}, \{4,6\}\},$$

where $\sigma = (4,5) \in S_{\mathbb{N}} = \mathrm{Aut}(K_{\mathbb{N},2})$.

k-symmetric function

- $R_{K_{\mathbb{N},k}} := \mathbb{C}[\![x_w \mid w \in \binom{\mathbb{N}}{k}]\!]$
- $S_{\mathbb{N}}$: the symmetric group

Define $\mathrm{Sym}^{(k)}$ to be the subring of $R_{K_{\mathbb{N},k}}^{S_{\mathbb{N}}}$ consisting of elements of finite degrees.

$$\downarrow$$

$X_{K_{\mathbb{N},k}}(G)$ belongs to $\mathrm{Sym}^{(k)}$. Since $X_{K_{\mathbb{N},k}}(G)$ is homogeneous of degree $|V(G)|$, $X_{K_{\mathbb{N},k}}(G)$ belongs to $\mathrm{Sym}^{(k)}$.

Note that $\mathrm{Sym}^{(1)}$ is the ring of symmetric functions and p_λ is the bases of $\mathrm{Sym}^{(1)}$.

Notation

Definition

Let $\{I_1, \ldots, I_n\}$ and $\{J_1, \ldots, J_n\}$ be two multisets consisting of elements in $\binom{\mathbb{N}}{k}$. Define an equivalence relation $\sim$ by $\{I_1, \ldots, I_n\} \sim \{J_1, \ldots, J_n\}$ if there exists $\sigma \in S_{\mathbb{N}}$ such that

$$\{I_1, \ldots, I_n\} = \{\sigma(J_1), \ldots, \sigma(J_n)\}$$

as multisets.

Definition

Let $\mathcal{P}_n^{(k)}$ denote the equivalence classes of such multisets discussed above and $\mathcal{P}^{(k)} := \bigsqcup_{n=1}^{\infty} \mathcal{P}_n^{(k)}$.

$$\downarrow$$

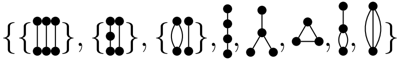
We can regard $\lambda \in \mathcal{P}_n^{(k)}$ as a k-uniform hyper-multigraph.

Example of $\mathcal{P}_n^{(k)}$

Example ($\mathcal{P}_3^{(1)}, \mathcal{P}_3^{(2)}$)

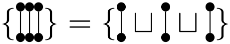
$$\mathcal{P}_3^{(1)} = \{\{1+1+1\}, \{1+2\}, \{3\}\}$$

$\mathcal{P}_3^{(2)} =$



We say that $\lambda \in \mathcal{P}^{(k)}$ is connected if it is connected as a hyper-multigraph.
$\implies \lambda = \lambda_1 \sqcup \cdots \sqcup \lambda_\ell$ in the usual manner.

Example



Linear basis for $\mathrm{Sym}^{(k)}$

Let $\lambda = \lambda_1 \sqcup \cdots \sqcup \lambda_\ell$. Then, we define the power sum k-fold symmetric function $p_\lambda = p_\lambda^{(k)} \in \mathrm{Sym}^{(k)}$ by

$$p_\lambda := \prod_{i=1}^{\ell} \sum_{\{I_1,\ldots,I_n\}\in\lambda_i} x_{I_1}\cdots x_{I_n}.$$

Example

$$p_{1+2} = \Big(\sum_{i\in\mathbb{N}} x_i\Big)\Big(\sum_{j\in\mathbb{N}} x_j^2\Big)$$

$$p_{\{\ \}} = \Big(\sum_{\substack{i,j,k\in\mathbb{N}\\ i\neq j\neq k}} x_{\{i,j\}}x_{\{j,k\}}\Big)\Big(\sum_{\substack{i,j\in\mathbb{N}\\ i\neq j}} x_{\{i,j\}}\Big)$$

When $k = 1$, p_λ is the usual power sum symmetric function.

The property of p_λ

Proposition

The set $\{p_\lambda\}_{\lambda \in \mathcal{P}^{(k)}}$ forms a linear basis for $\mathrm{Sym}^{(k)}$ over $\mathbb{C}$. In particular, $\mathrm{Sym}^{(k)}$ is freely generated as a $\mathbb{C}$-algebra by $\left\{ p_\lambda \in \mathcal{P}^{(k)} \;\middle|\; \lambda \text{ is connected} \right\}$.

Theorem

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}^{(k)}_{G_S}} p_\lambda$$

Recall that the weak complement expansion of $X_{K_{\mathbb{N},k}}$ is as following:

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{K_{\mathbb{N},k}}}(G_S)$$

Admissible

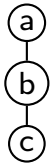
Definition (Admissible)

We say that $H \in \mathcal{P}_n^{(k)}$ is **admissible** by G if there exists a bijection $\varphi\colon V(G) \to E(H)$ such that for any $\{u, v\} \in E(G)$, it holds that $\varphi(u) \cap \varphi(v) \neq \emptyset$.

Example

We consider $G =$  and  $\in \mathcal{P}_3^{(2)}$.

The edge set of  is denoted by $\{\{1,2\},\{2,3\},\{3,4\}\}$. Then,  is admissible by , because there exists admissible bijection φ such that



$$\mapsto \{\{1,2\},\{2,3\},\{3,4\}\}$$

$$\varphi(a) = \{1,2\}, \varphi(b) = \{2,3\}, \varphi(c) = \{3,4\}.$$

Admissible

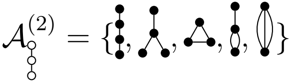
Definition ($\mathcal{A}_G^{(k)}$)

Let G be a connected graph with n vertices. We then define $\mathcal{A}_G^{(k)}$ as follows:

$$\mathcal{A}_G^{(k)} := \{H \in \mathcal{P}_n^{(k)} \mid H \text{ is admissible by } G\}.$$

If G is disconnected, and $G = G_1 \sqcup \cdots \sqcup G_\ell$ is its decomposition into connected components, we define $\mathcal{A}_G^{(k)} := \mathcal{A}_{G_1}^{(k)} \times \cdots \times \mathcal{A}_{G_\ell}^{(k)}$.

Example



Power sum expansion

Lemma

$$W_{\overline{K_{\mathbb{N},k}}}(G) = \sum_{\lambda \in \mathcal{A}_G^{(k)}} p_\lambda,$$

where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$ if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.

Proof.

When G is connected,

$$W_{\overline{K_{\mathbb{N},k}}}(G) = \sum_{\varphi \in \mathrm{Hom}^{\mathrm{w}}(G, \overline{K_{\mathbb{N},k}})} \prod_{v \in V(G)} x_{\varphi(v)} = \sum_{\lambda \in \mathcal{A}_G^{(k)}} p_\lambda.$$

$\square$

Power sum expansion

Theorem (Power sum expansion)

$$X_{K_{\mathbb{N},k}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}^{(k)}_{G_S}} p_\lambda,$$

where G_S is the spanning subgraph of G with edge set S.

Proof.

$$\begin{aligned} X_{K_{\mathbb{N},k}}(G) &= \sum_{S \subset E(G)} (-1)^{|S|} W_{\overline{K_{\mathbb{N},k}}}(G) \\ &= \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}^{(k)}_{G}} p_\lambda \end{aligned}$$

Kneser chromatic function distinguishes trees when $k = 2$

Main theorem

Hereinafter, we denote $\mathcal{A}_G$ and $\mathcal{P}_n$ by $\mathcal{A}_G^{(2)}$ and $\mathcal{P}_n^{(2)}$, respectively.

Theorem (Restatement)

$X_{K_{\mathbb{N},2}}(\bullet)$ is a complete invariant for trees.

Theorem (Power sum expansion)

$$X_{K_{\mathbb{N},2}}(G) = \sum_{S \subset E(G)} (-1)^{|S|} \sum_{\lambda \in \mathcal{A}_{G_S}} p_\lambda.$$

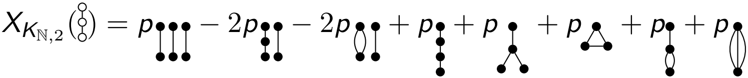
From power sum expansion, when G_1 and G_2 are trees,

$$X_{K_{\mathbb{N},2}}(G_1) = X_{K_{\mathbb{N},2}}(G_2) \Longrightarrow \mathcal{A}_{G_1} = \mathcal{A}_{G_2}.$$

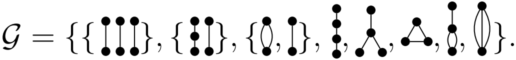
We consider the property of $\mathcal{A}_G$.

$\mathcal{A}_G$ from $X_{K_{\mathbb{N},2}}(G)$

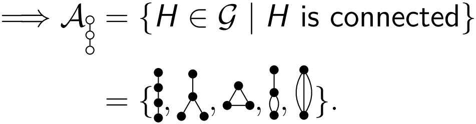
Example



Define



From power sum expansion, $\mathcal{G} = \bigcup_{S \subset E(G)} \mathcal{A}_{G_S}$ and when G is a tree, G_S is connected if and only if $G_S = G$.

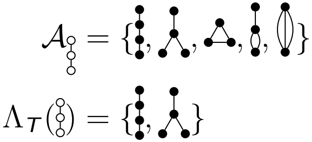


The properties of $\Lambda_T(G)$

We consider which $H \in \mathcal{A}_G$ best reflects the properties of G.
We define

$$\Lambda_T(G) := \{H \in \mathcal{A}_G \mid H \text{ is a tree with no multiedges}\}.$$

Example



Minimum rooted vertex sequence

- T: Tree
- $d(v)$: Degree of v
- $d_T(u, v)$: Distance from u to v

Definition (Minimum rooted vertex sequence)

Let $v = a_1$, and let $\{a_i\}_{i=1}^{n}$ be a vertex sequence of T that satisfies the following two conditions for any i and j:

- $d_T(v, a_i) \leq d_T(v, a_j)$.
- If $d_T(v, a_i) = d_T(v, a_j)$ and $i \leq j$, then $d(a_i) \leq d(a_j)$.

Then, we call $\{a_i\}_{i=1}^{n}$ a *minimum rooted vertex sequence*, and we refer to the degree sequence

$$(d(v), d(a_2), \cdots, d(a_n))$$

as *the minimum degree sequence of* v.

Example of minimum rooted vertex sequence

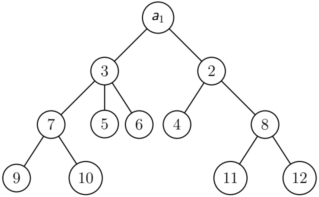


Figure: Minimum rooted vertex sequence

The degree sequence is

$$(2, 3, 4, 1, 1, 1, 3, 3, 1, 1, 1, 1).$$

Therefore,

$$r(T, a_1) = (2, 3, 4, 1, 1, 1, 3, 3, 1, 1, 1, 1).$$

Minimum leaf, minimum degree sequence

Definition (Minimum leaf)

A vertex v is said to be a *minimum leaf* if $r(T, v)$ is minimal in the lexicographical order among all vertices in T.

Definition (Minimum degree sequence)

We call minimum degree sequence of a minimum leaf the *minimum degree sequence of* T and denote it by $r(T)$.

Note that, generally, a minimum leaf is not unique, but the minimum degree sequence of a tree is always unique.

Example of minimum leaf and minimum degree sequence

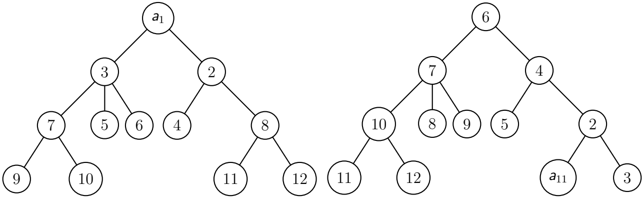


Figure: Minimum rooted vertex sequence

$$r(T, a_1) = (2, 3, 4, 1, 1, 1, 3, 3, 1, 1, 1, 1),$$
$$r(T, a_{11}) = (1, 3, 1, 3, 1, 2, 4, 1, 1, 3, 1, 1).$$

Thus,

$$r(T, a_{11}) \leq_{lex} r(T, a_1)$$

Example of minimum leaf and minimum degree sequence

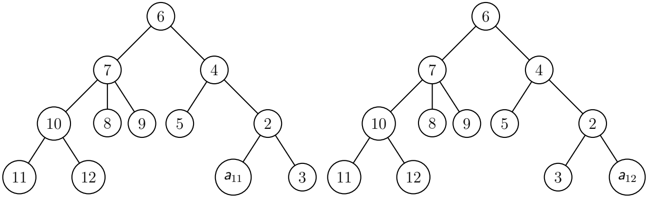


Figure: Minimum rooted vertex sequence

$$r(T, a_{11}) = r(T, a_{12}) = (1, 3, 1, 3, 1, 2, 4, 1, 1, 3, 1, 1).$$

Since a_{11} and a_{12} are the minimum leaves, the minimum degree sequence of T is

$$r(T) = (1, 3, 1, 3, 1, 2, 4, 1, 1, 3, 1, 1).$$

$\tilde{\Lambda}_T(G)$ is a complete invariant for trees

We define $\tilde{\Lambda}_T(G)$ as follows:

$$\tilde{\Lambda}_T(G) \coloneqq \{H \in \Lambda_T(G) \mid r(H) \text{ is the minimum by } \leq_{lex}\}.$$

Theorem (A)

Let $H \in \tilde{\Lambda}_T(G)$. Define H_0 as a tree which is obtained by removing any minimum leaf from H. Then, H_0 is isomorphic to G.

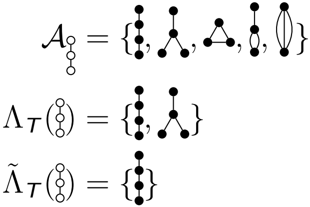
This implies $\tilde{\Lambda}_T(G)$ is a complete invariant for trees.
$\Longrightarrow$ This also shows $\mathcal{A}_G$ is a complete invariant for trees.

Example of Theorem

Theorem

Let $H \in \tilde{\Lambda}_T(G)$. Define H_0 as a tree which is obtained by removing any minimum leaf from H. Then, H_0 is isomorphic to G.

Example



The graph removing a minimum leaf from  is .

The proof of the main theorem

Theorem

$X_{K_{\mathbb{N},2}}(\bullet)$ is a complete invariant for trees.

Proof.

Let G_1 and G_2 be trees which holds $X_{K_{\mathbb{N},2}}(G_1) = X_{K_{\mathbb{N},2}}(G_2)$.
Then, $\mathcal{A}_{G_1} = \mathcal{A}_{G_2}$ and especially $\tilde{\Lambda}_T(G_1) = \tilde{\Lambda}_T(G_2)$.
Therefore, there exists H such that $H \in \tilde{\Lambda}_T(G_1) = \tilde{\Lambda}_T(G_2)$.
From Theorem A, $H_0 \simeq G_1$ and $H_0 \simeq G_2$. Therefore, $G_1 \simeq G_2$. $\square$

$$\boxed{X_{K_{\mathbb{N},2}}(G)} \xrightarrow[\text{expansion}]{\text{Power sum}} \boxed{\mathcal{A}_G} \longrightarrow \boxed{\tilde{\Lambda}_T(G)} \xrightarrow{\text{Theorem A}} \boxed{H_0 \simeq G}$$

Notation

- $\{a_i\}_{i=1}^{n}$: minimum rooted vertex sequence with $v = a_1$ as the root
- $d(a_i)$: The degree of a_i
- $r(G) = (d(a_1), \dots, d(a_n))$: Minimum degree sequence of G
- $r(G)_k = d(a_k)$

Definition

For any vertex $a_i \neq v$, a vertex a_{i_p} is called the parent of a_i with v as the root if a_{i_p} is adjacent to a_i and $i_p < i$.

Note that the parent of a_i is always uniquely determined.

Lemma A

Lemma (Lemma A)

For any $H \in \tilde{\Lambda}_T$,

$$r(H) \leq_{lex} (1, 2, r(G)_2, r(G)_3, \cdots, r(G)_n).$$

Proof.

We construct $H \in \Lambda_T(G)$ such that

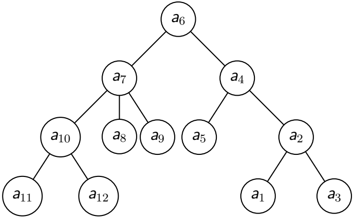
$$r(H) = (1, 2, r(G)_2, r(G)_3, \cdots, r(G)_n).$$

Define $\phi : V(G) \longrightarrow \binom{\mathbb{N} \cup \{0\}}{2}$ as follows:

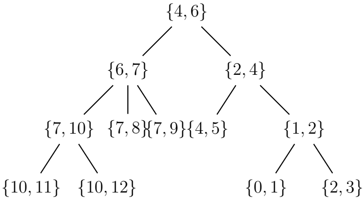
$$\phi(a_i) = \{i_p, i\},$$

where we define $1_p = 0$.

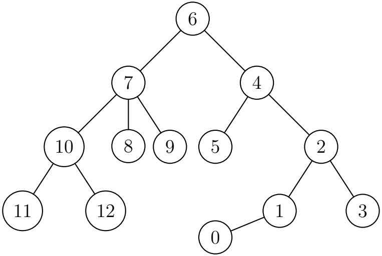
Example of $H \in \Lambda_T(G)$



(a) G

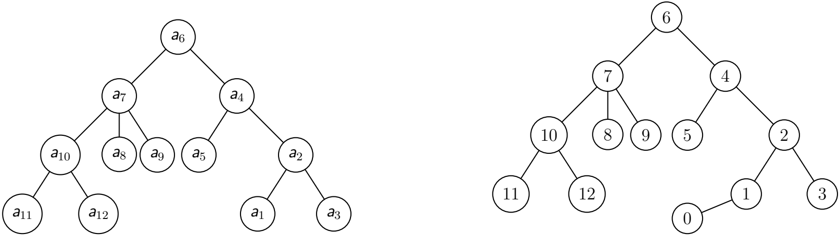


(b) $\mathrm{Im}(\phi)$



(c) H

The property of H



(a) G (b) H

$$r(G) = (1, 3, 1, 3, 1, 2, 4, 1, 1, 3, 1, 1)$$
$$r(H) = (1, 2, 3, 1, 3, 1, 2, 4, 1, 1, 3, 1, 1)$$

Proof of Lemma A.

Then, a tree H whose edge set is $\mathrm{Im}(\phi)$ is admisibble by G and

$$r(H) = (1, 2, r(G)_2, r(G)_3, \cdots, r(G)_n).$$

Theorem A

Theorem

Let $H \in \tilde{\Lambda}_T(G)$. Define H_0 as a tree which is obtained by removing any minimum leaf from H. Then, H_0 is isomorphic to G.

Proof.

Without loss of generality, we assume that $V(H) = \{0, 1, \ldots, n\}$, 0 is a minimum leaf, and $(0, 1, \ldots, n)$ is a minimum rooted vertex sequence. Then,

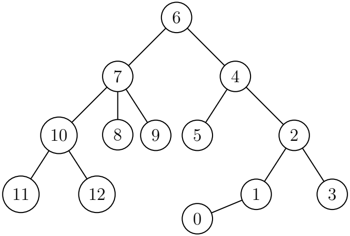
$$E(H) = \{\{i_p, i\} \mid 1 \leq i \leq n, i_p \text{ is a parent of } i\}.$$

Therefore,

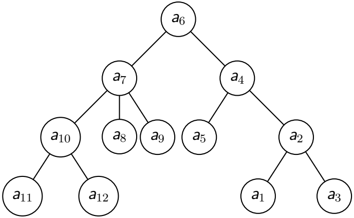
$$\begin{aligned} \tau : E(H) &\longrightarrow \{1, \ldots, n\} \\ \{i_p, i\} &\mapsto i \end{aligned}$$

is a bijection.

The proof of Theorem A



(a) H



(b) G

Figure: $\tau_H : a_i \mapsto i$

Since H is admissible by G, there exists a bijection $\phi : V(G) \longrightarrow E(H)$. Therefore, $\tau \circ \phi : V(G) \longrightarrow V(H_0)$ is also bijection. We show that $\tau_H = \tau \circ \phi$ is a graph isomorphism.

The property of τ_H

- $\tau_H(a_i) = i$
- $N_H(i), N_G(a_i)$: The neighbor of i in H and that of a_i in G, respectively.

Since ϕ is admissible by G,

$$\phi(N_G(a_i)) \subset \{\{x,y\} \in E(H) \mid \{i_p, i\} \cap \{x,y\} \neq \emptyset\}.$$

Applying τ to both sides, when $i_p \neq 0$,

$$\tau_H(N_G(a_i)) \subset N_{H_0}(i_p) \cup N_{H_0}(i).$$

If $i_p = 0$,

$$\tau_H(N_G(a_i)) \subset N_H(0) \cup N_{H_0}(i).$$

The proof of Theorem A

Proof.

We show $\tau_H(N_G(a_i)) = N_H(i)$ by induction on i. Since $H \in \tilde{\Lambda}_T(G)$, from Lemma A we obtain $r(H)_1 = |N_H(1)| = 1$ and $= r(H)_2 = |N_H(1)| \leq 2$. When $|V(G)| \geq 2$, $r(H)_2 = 2$ and this implies

$$\{\{x, y\} \in E(H) \mid 0 \in \{x, y\}\} = \{\{0, 1\}\}$$
$$\{\{x, y\} \in E(H) \mid 1 \in \{x, y\}\} = \{\{0, 1\}, \{1, 2\}\}.$$

Therefore, $\phi(v_1) = \{0, 1\}$ and $N_{H_0}(1) = \{0, 2\}$.

The proof of Theorem A

Proof.

Since ϕ is admissible by G,

$$\begin{aligned}\tau_H(N_G(a_1)) &\subset N_H(0) \cup N_{H_0}(1) \\ &= \{1, 2\}.\end{aligned}$$

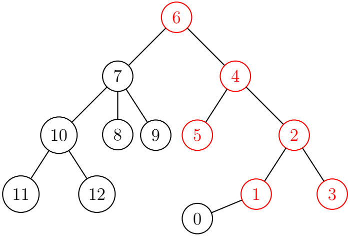
Because $\tau_H(a_1) = 1$ and $N_G(a_1) \neq \emptyset$,

$$\tau_H(N_G(a_1)) = \{2\} = N_{H_0}(1).$$

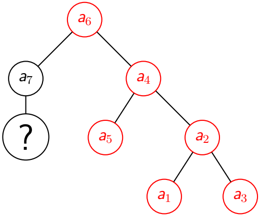
The proof of Theorem A

Proof.

Next, we assume that $\tau_H(N_G(a_i)) = N_{H_0}(i)$ holds for all i satisfying $1 \leq i \leq m < n$. Then, there exists $p_{m+1} < m+1$ such that $\phi(a_{m+1}) = \{p_{m+1}, m+1\}$.



(a) H



(b) G

Figure: $m = 6$

The proof of Theorem A

Similarly to before, since ϕ is admissible by G,

$$\tau_H(N_G(a_{m+1})) \subset N_{H_0}(p_{m+1}) \cup N_{H_0}(m+1).$$

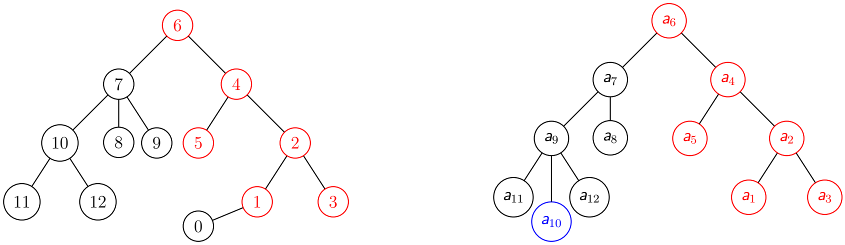
From the assumption, $\tau_H(N_G(a_{p_{m+1}})) = N_{H_0}(p_{m+1})$, especially $a_{m+1} \in N_G(a_{p_{m+1}})$. Therefore,

$$\tau_H(N_G(a_{m+1})) \cap \tau_H(N_G(a_{p_{m+1}})) = \emptyset$$

and

$$\tau_H(N_G(a_{m+1})) \subset N_{H_0}(m+1).$$

The proof of Theorem A



(a) H (b) G

Figure: If $\tau_H(N_G(a_7)) \subsetneq N_{H_0}(7)$

From Lemma A, $|N_{H_0}(m+1)| \leq r(G)_{m+1}$. If $\tau_H(N_G(a_{m+1})) \subsetneq N_{H_0}(m+1)$, then the degree sequence

$$(|N_G(a_1)|, \ldots, |N_G(a_{m+1})|, |N_G(a_{x_{m+2}})|, \ldots, |N_G(a_{x_n})|) <_{lex} r(G),$$

this is contradiction.

The proof of Theorem A

Proof.

Therefore, $\tau_H(N_G(a_{m+1})) = N_{H_0}(m+1)$. By induction, for all $i \in \{1, \ldots, n\}$, $\tau_H(N_G(a_i)) = N_{H_0}(i)$ and this implies $G \simeq H_0$. $\square$

Future work

- What types of graphs, other than trees, is $X_{K_{\mathbb{N},2}}(\bullet)$ a complete invariant for?
- For any k, do there exist graphs G_1 and G_2 such that $X_{K_{\mathbb{N},k}}(G_1) = X_{K_{\mathbb{N},k}}(G_2)$, but they are not isomorphic to each other?

The properties of $\Lambda_T(G)$

- $\mathcal{G}$: A set of all finite and simple graphs
- $L(G)$: The line graph of G

Proposition

Let G be a tree and define

$$\mathcal{L}_G := \{H \in \mathcal{G} \mid \text{There exists a homomorphism from } G \text{ to } L(H)\}.$$

Then, $\mathcal{L}_G$ is a complete imvariants for trees.

Question

Is there any other graph class for which $\mathcal{L}_G$ is a complete invariant?