Content

§1

Notation k: alg. cl. fld (char = p = 0)

$R = k[T_1 \cdots T_n]/I$

⇝ Spec R : affine variety

⇝ variety : Spec R の貼り合わせ.

ex: $\mathbb{A}^n_k = \operatorname{Spec} k[T_1 \cdots T_n]$; affine space

(Euclid space に対応)

$\mathbb{P}^n_k := \operatorname{Proj} k[T_0, \cdots, T_n]$: projective space

($\mathbb{A}^n_k$ の $(n+1)$ コの貼り合わせ)

Most important question

… To classify all projective variety

{ Graded ring R' を用いて Proj R' と書けるもの

invariant ⋯ dimension, <u>canonical divisor</u>

<u>Def</u> X : smooth variety.
$\quad \|$
$\quad \mathrm{Spec}\, R$

$$\underset{(\Omega_{R/k})}{\Omega_{X/k}} := \{ \textstyle\sum f_i dg_i \mid f_i, g_i \in \underset{(\in R)}{\mathcal{O}_X} \}$$

: loc. free of rank $= \dim X$

$\rightsquigarrow \omega_{X/k} = \Omega_{X/k}^{\wedge \dim X}$ ⋯ loc. free of rank $=1$
(=: invertible sheaf)

- $\Gamma \subset X$: prime divisor

$\Leftrightarrow \Gamma$: irreducible closed set of codim $=1$

- $D \in \mathrm{Div}\, X := \bigoplus_{\Gamma : \text{prime div}} \mathbb{Z}\Gamma$ is called a divisor
- $K(X)$: function field on X
- $f \in K(X)$, Γ : prime divisor

$\rightsquigarrow v_\Gamma(f)$: multiplicity of zero of f along Γ

- $\mathrm{Div}\,X/\sim \overset{1:1}{\longleftrightarrow} \mathrm{Pic}\,X = \{\text{invertible sheaf}\}$

$[D = \sum_\Gamma n_\Gamma \Gamma] \mapsto U \subseteq_{op} X \mapsto \mathcal{O}_X(D)(U)$

$$= \{0\} \cup \left\{ f \in k(X)^* \;\middle|\; \begin{array}{l} v_\Gamma(f) = -n_\Gamma \\ \text{for } {}^\forall \Gamma \text{ w/ } U \cap \Gamma = \phi \end{array} \right\}$$

$[K_X] \longleftarrow\!\shortmid\ \omega_{X/k}$

canonical divisor.

- Let $\mathcal{L} = \mathcal{O}_X(D)$

$H^0(\mathcal{L}, X) = \mathcal{L}(X) = \langle f_0, \dots, f_n \rangle_k$

$\rightsquigarrow \Phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^n_k$

$x \longmapsto [f_0(x) : \cdots : f_n(x)]$

: unique up to automorphism of $\mathbb{P}^n_k$

Def $\mathcal{L}$: very ample $\Leftrightarrow \Phi_{\mathcal{L}}$: cl. emb

ample $\Leftrightarrow \mathcal{L}^{\otimes m}$: v.a. for ${}^\exists m > 0$

Classification result

$\dim X = 1$

$$\kappa(X) = \begin{cases} -\infty \\ 0 \\ 1 \end{cases} \Leftrightarrow K_X \begin{cases} : \text{anti-ample} & \Rightarrow X \cong \mathbb{P}^1 \\ : \text{zero} & \Rightarrow X : \text{elliptic} \\ : \text{ample} \end{cases}$$

Kodaira dimension

$\Downarrow$

X : of general type

$\Downarrow$

$$\begin{cases} K_X : \text{v.a} \\ \quad \text{or} \\ \Phi_{\mathcal{O}(K_X)} : X \xrightarrow{2:1} \mathbb{P}^1 \end{cases}$$

$\dim X = 2$

$$\kappa(X) = \begin{cases} -\infty & \leftarrow \text{del Pezzo surfaces are here.} \\ 0 & \Rightarrow \cdots \\ 1 & \Rightarrow \cdots \\ 2 & \Rightarrow \cdots \end{cases}$$

§3 In what follow: $\dim X = 2$.

Def X, Y: smooth projective

- $f: Y \to X$: birational $\Leftrightarrow$ $k(Y) \cong k(X)$
- $\mathrm{Exc}(f) \subset Y$: reduced divisor (coeff = 1 or 0)

s.t. $\begin{cases} \cdot f \text{ induces } Y \setminus \mathrm{Exc}(f) \xrightarrow{\sim} X \setminus f(\mathrm{Exc}(f)) \\ \cdot f(\mathrm{Exc}(f)) \text{ are points.} \end{cases}$

$\omega_{Y/X} := \omega_{Y/k} \otimes f^* \omega_{X/k}$. loc free shf

$\left(\begin{array}{l} \omega_{X/k} \text{ is locally defined by elem } \alpha \in k(X) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \quad \downarrow \wr \\ f^*\omega_{X/k} \text{ is locally defined by } f^*\alpha \in k(Y) \end{array} \right)$

$= \mathcal{O}_Y(K_{Y/X})$

↑ relative canonical divisor.

Fact $^\forall E \subset \mathrm{Exc}(f)$: prime divisor on Y,

the multiplicity of $K_{Y/X}$ along E is independent

from the choice of $K_{Y/X}$

Def X: normal ($\dim X = 2$)

- X: klt $:\Leftrightarrow$ $^\forall Y$: normal. $^\forall f: Y \to X$: birational
  $^\forall E \subset \mathrm{Exc}(f)$: prime divisor on Y.
  (the mult of $K_{Y/X}$ along E) > -1.
- X: log del Pezzo $:\Leftrightarrow$ $-K_X$: ample & X: klt.
- X: log del Pezzo is of rank one
  $:\Leftrightarrow \mathrm{rk}_{\mathbb{Q}} (\mathrm{Div}\, X / \sim) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ ← main object.

Classification result.

- [Keel–McKernan '99] (char $k = p > 0$) gives the classification of log del pezzo surf of $\mathrm{rk} = 1$ except a bounded family ($= 41/42$-lc del Pezzo surface)
- [Lacini '24] ($p \neq 2, 3$) gives the classification of log del Pezzo surface of $\mathrm{rk} = 1$

§ Similarity between $p=0$ & $p>3$

In what follow, $\begin{cases} X : \text{log del Pezzo surf. of } rk=1 \\ \tilde{X} : \text{minimal resolution of } X \end{cases}$

<u>Def</u> X: normal (proj) surf.

$\cup$

$D = \sum_j a_j D_j$: boundary divisor $(0 \le a_i \le 1)$

- For $f : Y \to X$: birat. map from normal Y,

$\exists$ projection formula

$$K_Y + \sum_j a_j \underbrace{f_*^{-1} D_j}_{\text{strict transform}} + \sum_{E_i \subset \mathrm{Exc}(f)} e_i E_i = f^*(K_X + D)$$

$e(X, D; E_i) := e_i$ is called coefficient

w.r.t. $(X, D; E_i)$

<u>Fact</u>: $E_i \subset \tilde{X} \Rightarrow e(X, D; E_i) \ge 0$

• flush condition

Def $(X, D = \sum_{i=1}^{n} a_i D_i)$: as above

(X, D) is flush

$:\Leftrightarrow$ $^\forall E$: exceptional divisor / X,

$$e(X, D; E) < \min\{a_1 \dots a_n, 1\}$$

e.g. $(X, \emptyset)$ is flush.

Lem [KM. 8.0.4] (X, D) : flush pair

$\Rightarrow$ $^\forall t \in \mathrm{Sing}$ $(X, \lceil D \rceil)$ is plt at t.

e.g. [KM 8.3.6] $X \supset C$: prime divisor

$p \in C \cap (X \setminus \mathrm{Sing}\, X)$: simple cusp of C

$(x^p = y^q,\ p.q \geq 2.\ \mathrm{GCD}(p.q) = 1)$

then (X, aC) is flush at p $\Leftrightarrow$ $a \leq \frac{4}{5}$

- tigen.

<u>Def</u> (X, Δ) : log pair.

$f: Y \to X$: bir. morph.

Then prime divisor $E \subset Y$ is tigen

$\Leftrightarrow$ $\exists$ $\alpha \geq 0$: $\mathbb{Q}$-divisor on X s.t.

$$\begin{cases} \cdot\ K_X + \Delta + \alpha \equiv 0 \\ \cdot\ e(X, \Delta + \alpha; E) \geq 1 \end{cases}$$

e.g. If $|-K_X| \neq \phi$. (e.g. X is Du Val del Pezzo)

then (X, ϕ) has a tigen

- Hunt step ([KM, 8.2.5])

… a sequense of certain Sarkisov links

- Input : X: log del Pezzo surface of rk = 1

  without tiger in $\tilde{X}$ ($\Rightarrow$ X: not Du Val)

  $(X_0, \Delta_0) = (X, \phi)$

<u>i-th step</u> (X_i, Δ_i) : log pair without tiger in $\widetilde{X_i}$

s.t. $-(K_{X_i} + \Delta_i)$: ample

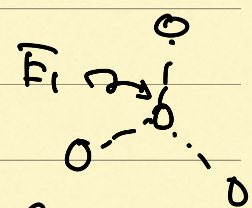
(1) extract excep. div. E_{i+1} in $\tilde{X}_i$

with highest coeff. (+α condition)

set $x_i :=$ center of E_{i+1} on X_i $(C_{X_i}(E_{i+1}))$

+α : ① when x_i is cyclic.

choose E_1 as not (-2)-curve



② when x_i is non-cyclic.

choose E_1 as the center of the fork

$$\begin{array}{ccccc}
 & & P^{\circ}_{i+1}+e_iE_{i+1} & & \\
 & & \uparrow & \text{perturb.} & \\
E_{i+1} & & (T_{i+1}, P_{i+1}) & \rightsquigarrow & (T_{i+1}, P'_{i+1}) \\
\downarrow & & f_i \downarrow \ \begin{array}{l}\text{extraction}\\ \text{of } E_i\end{array} & & \begin{array}{l}\text{contraction}\\ \text{of } R\end{array} \downarrow \ \pi_{i+1} \\
x_i & & (X_i, \Delta_i) & & (X_{i+1}, \Delta_{i+1})
\end{array}$$

Take P_{i+1} s.t. $K_{T_{i+1}} + P_{i+1} = f_i^*(K_{X_i} + \Delta_i)$

$NE(T_{i+1})$ is geaerated by 2 rays

$\mathbb{R}_{\geq 0}[E_i]$ & R: $K_{T_{i+1}}$-negathe

Take π_{i+1} as the contraction of R

(2) choose $0 < \delta \ll 1$ & $\tau > 0$ s.t.

$P'_{i+1} := (1+\tau)(P^{\circ}_{i+1} + (e_i + \delta)E_{i+1})$ satisfies

$K_{T_{i+1}} + P'_{i+1}$ is R-trivial

$\rightarrow$ define Δ_{i+1} as

$K_{T_{i+1}} + P'_{i+1} = \pi_{i+1}^*(K_{X_{i+1}} + \Delta_{i+1})$

$\rightsquigarrow$ <u>Output</u> (X_{i+1}, Δ_{i+1}) : log pair. without tiger in $\widetilde{X_{i+1}}$ s.t. $-(K_{X_{i+1}} + \Delta_i)$: klt

<u>Fact</u> $P_{i+1} \rightsquigarrow P'_{i+1}$ increase coefficients

$\rightsquigarrow$ limit step often preserve flushness

(exception: singularity of $\ulcorner \Delta_i \urcorner$ in $(X_i \setminus \mathrm{Sing}\, X_i)$)

Thm ([KM 23.2] [Lacini. Thm 6.2], N)

Suppose that X has a tiger E_1 in $\widetilde{X}$.

(1) If E_1 is exceptional,

then we can replace E_1 as an excep. tiger $\widetilde{X}$ with max. coefficient $(+\alpha)$ s.t.

<u>0-th step</u> gives one of the following:

(a) π_1 is a $\mathbb{P}^1$-fibration and

E_1 is a section or a bisection

(b) $(X_1, C_1 := \mathrm{Supp}\,\Delta_1)$ is a plt pair.

s.t. $-(K_{X_1}+C_1)$: nef

(c) X_1 : Du Val del Pezzo surface.

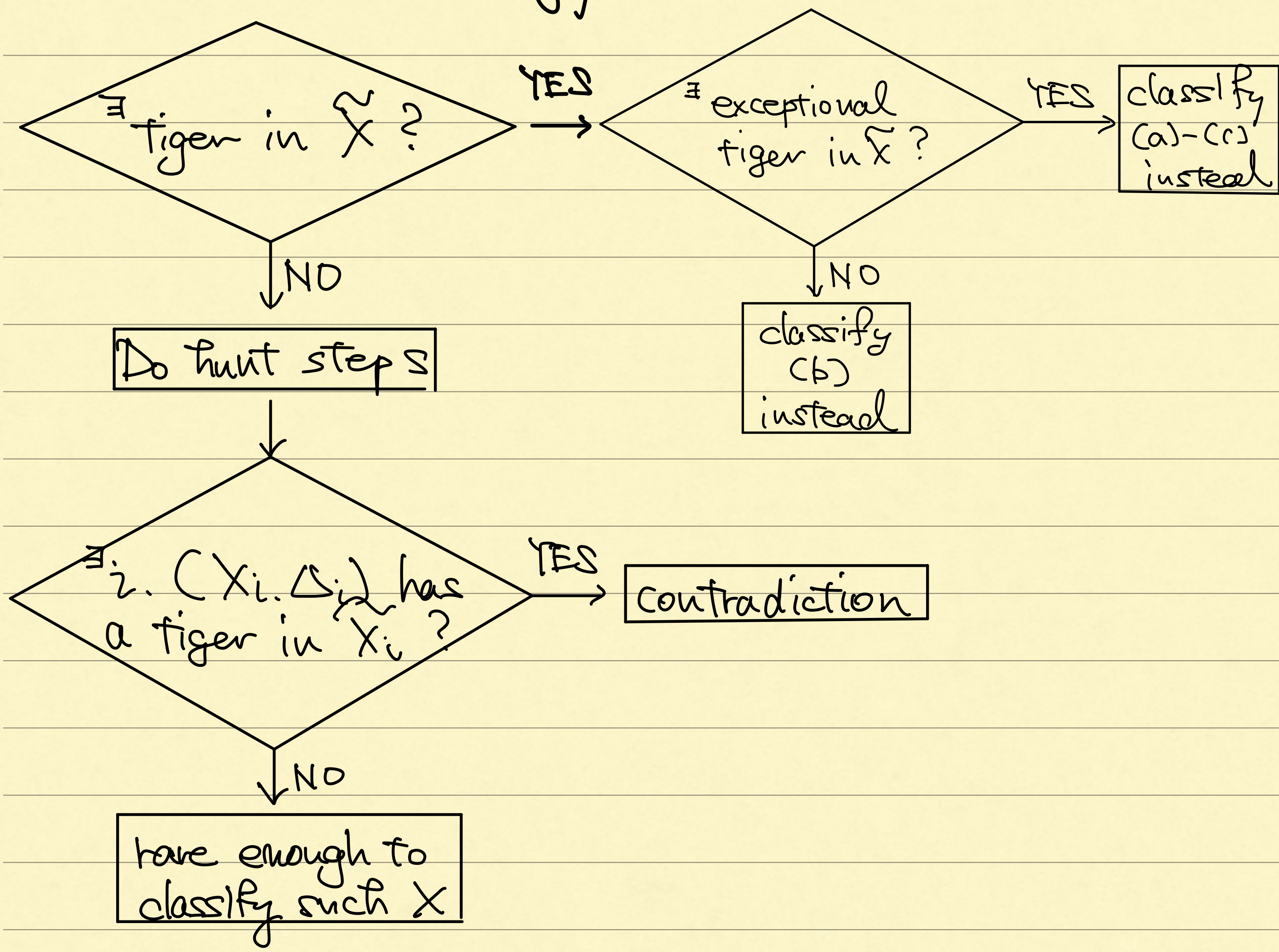
s.t. $\begin{cases} \cdot\ \mathrm{Supp}\,\Delta_1 =: C_1 \in |-K_{X_1}| \\ \cdot\ C_1 \cap \mathrm{Sing}\,X_1 = \phi \end{cases}$

(2) If $^{\forall}$ tiger in $\tilde{X}$ is non-exceptional,

then $^{\exists} C \subset X$: prime divisor s.t.

(X, C) : plt & $-(K_X + C)$: nef (Case (b))

General strategy



§ Difference between $p=0$ & $p>3$

- "$\pi_1(X \setminus \mathrm{Sing}\, X)$"

[KM] assumes $\pi_1(X \setminus \mathrm{Sing}\, X) = 1$

when X has no tiger

[Lacini] avoid this assumption by using

the classification of Du Val del Pezzo surf

of $rk = 1$ w/ singular member of $|-K|$

e.g X : Du Val del pezzo surface with $2A_4$-sing.

$\cup$

$C \in |-K_X|$. $C \cap \mathrm{Sing}\, X = \phi$

$\begin{cases} p \neq 5 \Rightarrow C \text{ is nodal. } \exists \text{ exactly two such } C \\ p = 5 \Rightarrow C \text{ is cuspidal } \exists \text{ exactly one such } C. \end{cases}$

- Bogomolov bound

$p=0 \Rightarrow \displaystyle\sum_{\tau \in \mathrm{Sing}\, X} \frac{\mathrm{index}(\tau)-1}{\mathrm{index}(\tau)} \leq 3$

this bond is false in $p > 0$

e.g. ch $= 5 \Rightarrow \exists X$: log del Pezzo surf of rk $= 1$

with 5 singular pt $2A_4 + A_1 + [3] + [5]$

$$\sum_{t \in \mathrm{Sig} X} \frac{\mathrm{index}(t) - 1}{\mathrm{index}(t)} = 2 \cdot \frac{4}{5} + \frac{1}{2} + \frac{2}{3} + \frac{4}{5}$$

$$= \frac{107}{30} > 3$$

Result

Thm [Lac]:

(the list of log del Pezzo surface of rk $= 1$ in $p > 5$)

is the same as

(the list of — " — in $p = 0$)

Moreover, $\forall$ log del Pezzo surface of rk $= 1$ in $p > 5$

is log liftable to char $= 0$.