加群圏でのねじれ理論と近似理論

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[Kato] Kato, K.: Morphisms represented by monomorphisms. J. Pure Appl. Algebra 208(1), 261–283 (2007)

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§1 射の安定同値変形

$\mathcal{A}$: abelian category with enough projectives.

$\cup$ proj $\mathcal{A}$. proj. objs of $\mathcal{A}$.

Motivation : 与えられた射を、ある程度の差を許容して、
「良い性質」(e.g. 全射、単射、同型)を持つ射に変形したい。
→ここでは、射影因子の差を許す。

すなわち、f : morphism in $\mathcal{A}$ を stable category $\underline{\mathcal{A}} := \mathcal{A}/\mathrm{proj}\,\mathcal{A}$ では同値になるよう変形する。

$\boxed{\text{Ex}}$ $f : X \to Y$ in $\mathcal{A}$. Take $P_Y \xrightarrow{p} Y$: epimorphism.

Then $(f, p) : X \oplus P_Y \twoheadrightarrow Y$: epi.

Note
$$\begin{array}{ccc}
\underline{X} & \xrightarrow{\underline{f}} & \underline{Y} \\
\underline{\binom{1}{0}}\downarrow\cong & \circlearrowleft & \cong\| \ \underline{1} \\
X \oplus P_Y & \xrightarrow[\underline{(f,p)}]{} & \underline{Y}
\end{array} \quad \text{in } \underline{\mathcal{A}} \qquad \left(\underline{(f,p)} \cong \underline{f} \text{ in } \underline{\mathcal{A}} \right)$$

Def [ABr, Kato]

$f : X \to Y$: morph. in $\mathcal{A}$.

f : represented by monomorphisms (rbm)

$:\Leftrightarrow$ $\exists P, Q \in \operatorname{proj}\mathcal{A}$, $\exists s, t, u$: morphs in $\mathcal{A}$ s.t.

$\begin{pmatrix} f & s \\ t & u \end{pmatrix} : X \oplus P \to Y \oplus Q$: monomorphism (in $\mathcal{A}$)

$\boxed{\text{Ex}}$ $f : X \to Y$ in $\mathcal{A}$

(1) f : mono $\Rightarrow$ f : rbm

(2) $X = P \in \operatorname{proj}\mathcal{A}$ $\Rightarrow$ f : rbm. ($\because$ $\begin{pmatrix} f \\ 1 \end{pmatrix} : P \to Y \oplus P$: mono.)

More generally, X : torsionless ($:\Leftrightarrow \exists X \hookrightarrow P \in \operatorname{proj}\mathcal{A}$)

$\Rightarrow$ f : rbm.

(3) X : not torsless, Y : torsless $\Rightarrow$ f : <u>NOT</u> rbm

($\because$) $\exists Y \hookrightarrow Q \in \operatorname{proj}\mathcal{A}$

If f : rbm, then $\exists P', Q' \in \operatorname{proj}\mathcal{A}$, $X \oplus P' \hookrightarrow Y \oplus Q'$.

So $X \hookrightarrow X \oplus P' \hookrightarrow Y \oplus Q' \hookrightarrow Q \oplus Q' \in \operatorname{proj}\mathcal{A}$.

$\therefore$ X : torsless ※

$f : X \to Y$.

X \ Y	trl	not trl
trl	rbm	rbm
not trl	not rbm	$\boxed{??}$ ← このときどうなる？？

⟶ <u>module category</u> の場合を考える. 以下, R : noeth. ring

{right R-modules} $\supset$ {fin. gen. R-mods} $\supset$ {fin. gen. proj. R-mods}

$\qquad\|\qquad\qquad\qquad\|\qquad\qquad\qquad\|$

Mod R $\qquad$ mod R $\qquad$ proj R

$\boxed{\text{Ex}}$ $R = k[[X,Y]]/(XY)$: 1-dim (A_1) singularity.

(1) $X = R/(x^2)$ $\quad Y = R/(x^2,y)$ $\quad f : X \to Y$, $\quad X = R/(x^2) \overset{\text{nat}}{\twoheadrightarrow} R/(x^2,y) = Y$

($\to X, Y$: not torsless)

$t : X = R/(x^2) \to R$, $\quad t = \hat{y} : \bar{a} \mapsto ay$

Then $\binom{f}{t} : X \to Y \oplus R$: inj.

$\left(\binom{f}{t}(\bar{a}) = 0 \Rightarrow \begin{array}{l} a \in (x^2, y) \\ \wedge \\ a \in (0 :_k y) = (x) \end{array} \Rightarrow a \in (x^2) \right)$

$\therefore$ f : rbm

(2) $X = R/(x^2)$ $\quad Z = R/(x,y)$ $\quad g : X \to Z$, $\quad X = R/(x^2) \overset{\text{nat.}}{\twoheadrightarrow} R/(x,y) = Z$

($\to X, Z$: not torsless)

Then g : NOT rbm.

In fact, if g : rbm, then $\exists \begin{pmatrix} g & * \\ * & * \end{pmatrix} : X \oplus R^{\oplus} \hookrightarrow Z \oplus R^{\oplus}$.

May assume $\exists\ {}^t(g, t_1, t_2, \ldots, t_a) : X \hookrightarrow Z \oplus R^{\oplus a}$.

Note $\operatorname{Hom}_R(X, R) = \operatorname{Hom}_R(R/(x^2), R) \cong (y)$

$(yr\text{倍}) \leftarrow yr$

$\therefore \exists r_i \in R$, $t_i = yr_i : X \to R$ $(1 \le \forall i \le a)$

Then $\operatorname{Ker}({}^t(g, t_1, \ldots, t_a) : X \to Z \oplus R^{\oplus a})$

$= \operatorname{Ker} g \cap \operatorname{Ker}(yr_1) \cap \cdots \cap \operatorname{Ker}(yr_a) \ni \bar{x} \neq \bar{0}$ $(\in X = R/x^2)$

$\#$

Thm [Kato] R : comm. noeth. ring.

Assume R : generically Gorenstein ($:\Leftrightarrow \forall \mathfrak{p} \in \operatorname{Ass} R$, $R_{\mathfrak{p}}$: Gorenstein).

Then $\forall f : X \to Y \in \operatorname{mod} R$,

f : rbm $\Leftrightarrow$ $\operatorname{Ker} f$: torsionless

$\boxed{\text{Ex}}$ In the above example,

(1) $\mathrm{Ker}(f: R/(x^2) \twoheadrightarrow R/(x^2,y)) = (x^2,y)/(x^2) \cong (y) \hookrightarrow R$. ∴ torsless.

(2) $\mathrm{Ker}(g: R/(x^2) \twoheadrightarrow R/(x,y)) = (x,y)/(x^2) \hookleftarrow k$ ∴ not torsless.

Some remarks on rbm

① 表現論的側面 (上の Thm の逆)

Thm [Kato] R: comm. noeth.

$\mathrm{Ker} f$: torsless なる $\forall f$ in $\mathrm{mod} R$ が rbm $\Rightarrow$ R: gen. Gor.

② Stable isom. との関係 stable isom. :$\Leftrightarrow$ rep. by isom $\Leftrightarrow$ isom. in $\underline{\mathrm{mod}}R$.

Prop [O] f in $\mathrm{mod}R$ Then

f: stab. isom. $\Leftrightarrow$ $\underline{\mathrm{Ker}} f$: proj. & $\mathrm{Tr} f$: rbm.

stable kernel Auslander–Bridger transpose

③ Applications [ABr] : { • Spherical filt. thm (cf. [O, 2023, PAMS]) • Origin extension (below) (see also [Kato, 1999, CA]) }

Thm (origin ext.) Let $M \in \mathrm{mod}R$ and $n \geq 0$.

Assume $\mathrm{grade}\, \mathrm{Ext}_R^i(M,R) \geq i$ $(1 \leq \forall i \leq n)$.

Then $\exists$ $0 \to X \to M \oplus P \to Y \to 0$: ex in $\mathrm{mod}R$ s.t.

$P \in \mathrm{proj}R$, $\mathrm{Ext}^{1\sim n}(X,R) = 0$, $\mathrm{proj.dim}\, Y \leq n$

$\rightsquigarrow$ refine MCM approx. over Gor. rings. (FPD hull → [O, 2024, JA])

Three approaches to rbm

- stable mod. theory [ABr, Kato, O] §3
- homotopy cat. theory [Kato] §3
- torsion theory [Kato appendix by Takashima] §2

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§2 Lambek torsion theory

(cf. [Hoshino, 1992, Osaka J.]
[Hoshino − Takashima, 1994, Osaka J.]
[Hoshino, 1995, Osaka J.])

R: ring. $M, N \in \mathrm{Mod}R$ $\quad \varepsilon_M^N := \bigcap_{f \in \mathrm{Hom}_R(M,N)} \mathrm{Ker} f \subset M$: R-submod.

$$= \{ x \in M \mid \forall f : M \to N,\ f(x) = 0 \}$$

Prop [Kato (append.), 0] R: noeth. $f : X \to Y$ in modR.
Then f: rbm $\Leftrightarrow$ $\mathrm{Ker} f \cap \varepsilon_M^R = 0$

$\tau_M := \varepsilon_M^{E(R)}$: Lambek torsion submodule of M.

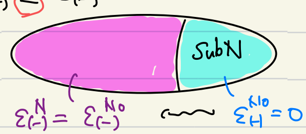
Note $\tau_M \subset \varepsilon_M^R$ (because $R < E(R)$). When "$=$" ??

Prop $\mathcal{X} \subset \mathrm{Mod}R$: closed under quotients (i.e. $M \in \mathcal{X}$, $M \twoheadrightarrow M' \Rightarrow M' \in \mathcal{X}$)
$N_0 \subset N$ in ModR. Then TFAE.

(1) $\forall M \subset N$ with $M \in \mathcal{X}$, $\varepsilon_M^{N_0} = 0$

(2) $\forall M \in \mathcal{X}$, $\varepsilon_M^N = \varepsilon_M^{N_0}$.

$\varepsilon_{(-)}^N \subseteq \varepsilon_{(-)}^{N_0}$

$\mathcal{X})$ 

(proof) $\underline{(1) \Rightarrow (2)}$ $M \in \mathcal{X}$. [WTS $\forall g : M \to N$, $\varepsilon_M^{N_0} \subset \mathrm{Ker}\, g$]

$N \supset \mathrm{Im}\, g \cong M/\mathrm{Ker}\, g \underset{\text{quo. cl.}}{\in} \mathcal{X}$. $\quad \therefore \varepsilon_{\mathrm{Im} g}^{N_0} = 0$.

$$\begin{array}{ccc} \varepsilon_M^{N_0} & \longrightarrow & \cancel{\varepsilon_{\mathrm{Im} g}^{N_0}} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\hat{g}} & \mathrm{Im}\, g \end{array} \qquad \therefore g(\varepsilon_M^{N_0}) = 0$$

//

Lem $M_0 \subset M$ in ModR. $E \in \mathrm{Inj}R$. Then $\varepsilon_{M_0}^E = M_0 \cap \varepsilon_M^E$

(proof) "$\supset$" のみ, $x \in$ (RHS)

Let $g : M_0 \to E$.

$$\begin{array}{ccccc} 0 & \to & M_0 & \overset{\subseteq}{\to} & M \\ & & g \downarrow & \swarrow h & \\ & & E & & \end{array} \qquad x \in \varepsilon_M^E \rightsquigarrow \underset{g(x)}{\overset{\|}{h(x)}} = 0$$

//

↳ In non comm. case, replace (1) with "R: (Auslander's) 1-Gorenstein".

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\quad R:\text{comm. noeth.}\quad \text{TFAE.}\\ (1)\ R:\text{gen. Gor.}\\ (2)\ \forall M\subset E(R)\ :\text{fin. gen. sub } R\text{-mod},\quad \varepsilon_M^R=0\ \ (\Leftrightarrow\ M:\text{torsless})\\ (3)\ \forall M\in \mathrm{mod}R,\quad \varepsilon_M^R=\varepsilon_M^{E(R)}.\end{array}}$$

(sketch) $\underline{(1)\Rightarrow(2)}$ $\mathrm{Ass}M\subset \mathrm{Ass}E(R)=\mathrm{Ass}R$. $\therefore\ \forall \mathfrak{p}\in\mathrm{Ass}M$, $R_\mathfrak{p}$: Artin. Gor.

$\therefore\ \mathrm{Ass}(\varepsilon_M^R=\mathrm{Ker}(M\to M^{**}))=\phi$. $\therefore\ \varepsilon_M^R=0$.

$\underline{(2)\Leftrightarrow(3)}$ follows from above $\underline{\text{Prop}}$ ($\mathcal{X}:=\mathrm{mod}R$). //

$\boxed{\text{Proof of Kato's thm (Takashima)}}$ R: gen. Gor. $f:X\to Y$ $\in$ $\mathrm{mod}R$.

$$f:\text{rbm}\Leftrightarrow 0=\mathrm{Ker}f\cap\varepsilon_X^R\ \underset{\text{Thm}}{=}\ \mathrm{Ker}f\cap\varepsilon_X^{E(R)}\ \underset{E(R):\text{inj}}{=}\ \varepsilon_{\mathrm{Ker}f}^{E(R)}\ \underset{\text{Thm}}{=}\ \varepsilon_{\mathrm{Ker}f}^R\quad //$$

§3 ホモトピー圏への埋め込みと単射表現性の高階化

R: noeth. $K(\mathrm{proj}R):=\{\text{cpxs of proj. mods}\}/\{\text{null cpxs}\}$ $\ni$

$$\left(\begin{tikzcd}\cdots \arrow[r] & X^{i-1} \arrow[r] \arrow[dl] \arrow[d, equal] & X^{i} \arrow[r] \arrow[dl] \arrow[d, equal] & X^{i+1} \arrow[r] \arrow[dl] \arrow[d, equal] & \cdots \arrow[dl] \\ \cdots \arrow[r] & X^{i-1} \arrow[r] & X^{i} \arrow[r] & X^{i+1} \arrow[r] & \cdots\end{tikzcd}\right)$$

: triangulated category

$$\boxed{\begin{array}{l}\underline{\text{Thm}}\ (1)\ [\text{Kato (2001)}]\quad F:\underline{\mathrm{mod}}R\hookrightarrow K(\mathrm{proj}R)\\ \qquad\qquad X\mapsto F_X:=(\cdots\to P^{-1}\to P^{0}\to P^{1}\to P^{2}\to\cdots)\\ \qquad\qquad\qquad\qquad\qquad \text{ex}\qquad\quad E^1\qquad E^2\qquad E^3\\ \qquad\qquad M\to M^{**}\qquad (E^i=\mathrm{Ext}^i(\mathrm{Tr}M,R))\\ (2)\ [\text{Kato (2007)}]\quad f:X\to Y\ \in\ \mathrm{mod}R.\\ \qquad (\leadsto F_X\xrightarrow{Ff}F_Y\to C_f\to F_X[1]\ :\ \text{triangle}\ \in\ K(\mathrm{proj}R))\\ \qquad\qquad\qquad\qquad\quad \text{mapping cone}\\ \quad\text{Then}\quad f:\text{rbm}\Leftrightarrow H^{-1}(C_f)=0.\end{array}}$$

$\boxed{\text{Ex}}$ $R=k[X,Y]/(X^2,XY,Y^2)$. $\underline{\text{Compute}}$ $F_k\in K(\mathrm{proj}R)$.

$$\cdots\to \underset{P^{-2}}{R^4}\xrightarrow{\begin{pmatrix}x&y&&\\&&x&y\end{pmatrix}}\underset{P^{-1}}{R^2}\xrightarrow{(x\ y)}\underset{P^{0}}{R}\to k\to 0\ :\text{free resol. of } k.$$

$$\cdots \to R^4 \xrightarrow{\begin{pmatrix} x & y & & \\ & & x & y\end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \xrightarrow{\binom{x}{y}} R^2 \to \mathrm{Tr}k \to 0 \ : \text{free resol. of } \mathrm{Tr}k.$$
$$P^{2*} \qquad P^{1*} \qquad P^{0*} \qquad P^{1*}$$

$$\therefore F_k = \left(\cdots \to \underset{-2}{R^4} \xrightarrow{\begin{pmatrix} x & y & & \\ & & x & y\end{pmatrix}} \underset{-1}{R^2} \xrightarrow{(x\ y)} \underset{0}{R} \xrightarrow{\binom{x}{y}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x & \\ y & \\ & x \\ & y\end{pmatrix}} \underset{2}{R^4} \to \cdots\right)$$

<u>Def</u> (1) $M \in \mathrm{mod}\,R$. n-torsionfree $:\Leftrightarrow H^i(F_M) = 0 \quad 0 \le i \le n-1$.

(2) $f : X \to Y$ in $\mathrm{mod}R$.

$\underline{\mathrm{Ker} f} := H^{-1}(\tau_{\le -1} C_f)$ hard truncation $\tau_{\le n}F = (\cdots \to F^{n-1} \to F^n \to 0 \to 0 \to \cdots)$

$\underline{\mathrm{Cok} f} := H^0(\tau_{\le 0} C_f)$: stable kernel & stable cokernel

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<u>Rmk</u> (1) 1-torsfree = torsless 2-torsfree = reflexive

(2) stable kernel (coker.) $\Rightarrow$ weak kernel (coker.) of $\underline{f}$ in $\underline{\mathrm{mod}}\,R$.

<u>Def</u> [O] (higher rbm) $n \ge 0$.

$f : X \to Y$ in $\mathrm{mod}R$: (T_n) $:\Leftrightarrow H^i(C_f) = 0 \quad -1 \le \forall i \le n-2$

<u>Thm</u> [O] $f : X \to Y$ in $\mathrm{mod}R$. Consider the following conds.

(1) $f : (T_n)$

(2) $\exists\ 0 \to X \xrightarrow{\binom{f}{t}} Y \oplus P \to Z \to 0$: ex s.t $\binom{f}{t}^*$: surj $Z : (n-1)$-torsfree.

(2') $\forall$ ______ : ex with $\binom{f}{t}^*$: surj , Z : ______

(3) $\underline{\mathrm{Ker} f}$: n-syzygy (4) $\underline{\mathrm{Ker} f}$: n-torsfree

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2') $\underset{(b)}{\overset{(a)}{\rightleftarrows}}$ (4) always hold. Moreover, consider

(2') $\Rightarrow$ (3), (4) $\Rightarrow$ (3)

(a) $\mathrm{grade}\,\mathrm{Ker}\left(\mathrm{Ext}^1(f,R) : \mathrm{Ext}^1(Y,R) \to \mathrm{Ext}^1(X,R)\right) \ge n-1$

(b) ______ $\ge n$.

Then $\rightleftarrows$ hold.

$n=2$

$\boxed{\text{Ex}}$ R: comm. $f: X \to Y \in \operatorname{mod} R$. Suppose $\operatorname{grade} \operatorname{Ext}^1(Y,R) \geq 1$
(e.g. $\operatorname{grade} Y \geq 1$)

Then

- f: (T_2) $\Longrightarrow$ $\underline{\operatorname{Ker} f}$: reflexive.
- $\underline{\operatorname{Ker} f}$: ref & $\operatorname{grade} \operatorname{Ext}^1(Y,R) \geq 2$ $\Longrightarrow$ f: (T_2).

$\boxed{\text{Ex}}$ $R = k[[x,y]]/(xy)$ $\quad f: \underset{X}{\underset{\shortparallel}{R/(x^2)}} \xrightarrow{\text{nat}} \underset{Y}{\underset{\shortparallel}{R/(x^2,y)}}$: rbm $(=(T_1))$.

$$\begin{tikzcd}
0 \arrow[r] & \varepsilon_X^R \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X^{**} \arrow[d] \\
0 \arrow[r] & \varepsilon_Y^R \arrow[r] & Y \arrow[r] & \cancel{Y^{**}}
\end{tikzcd}$$

$\varepsilon_X^R = (x)/(x^2) \not\cong Y$.

$\therefore$ By $\underline{\text{Thm}}$, f: not (T_2).

Application

> $\underline{\text{Thm}}$ [O] (higher origin ext.)
> $n, m \geq 0$. $\forall M \in \operatorname{mod} R$ with $\operatorname{grade} \operatorname{Ext}^i(M,R) \geq i+m$ $(1 \leq \forall i \leq n)$,
> $\exists\ 0 \to X \to M \oplus P \to Y \to 0$: ex $\quad \operatorname{Ext}^{1\sim n}(X,R) = 0$
> $\operatorname{proj.dim} Y \leq n$ & Y: m-torsfree.