2次元特異点と純非分離拡大

Part A Intro (1 hour)

§1. Why singularity?
§2. klt 特異点

Part B Graph (1.5 hour) §3,4,5
Part C Main Thm (30 min.)

Part A Intro
§1. Why singularity?

K : 体 ($^\forall$chara)
$R = K[x_1, \cdots, x_n]/I$; int. dom. of fin. $/K$

$\rightsquigarrow$ $\mathrm{Spec}(R)$

- Variety $/K$ … $\mathrm{Spec}(R)$ やその貼り合わせでとらえられる図形

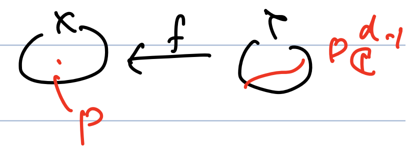
<u>eg.</u> $\mathrm{Spec}\left(K[x_1, \cdots, x_n]/(f) \right) \longleftrightarrow Z(f) := \{ (a_1, \cdots, a_n) \in K^n \mid f(a_1, \cdots, a_n) = 0 \}$

$\boxed{\text{① 分類}}$

<u>Central Problem in alg. geom.</u>
Classify all smooth proj var/$\mathbb{C}$

Difficulty

- $\exists$ Too many smooth proj. var.



f : isom. out side p

· $f^{-1}(p) = \mathbb{P}^{d-1}$

eg. 1

X : smooth proj. var / $\mathbb{C}$, $p \in X$: (closed) pt.

$\rightsquigarrow$ $f : Y \to X$: blow up along $\{p\}$.

$\rightsquigarrow$ $Y \in$ sm proj. var ② $Y \not\cong X$ (dim ≥ 2)

解決策

「何かの bl-up」とはかけないものの[illegible]分類することを考えよう.

$\boxed{\text{◎ MMP}}$

Fact 2 (minimal model program in dim 2)

Y : sm proj. var / $\mathbb{C}$, dim $= 2$

$\rightsquigarrow$ $\exists X$: sm proj. var / $\mathbb{C}$, dim $= 2$

② seq. of bl-up $X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n = Y$

s.t. X = relatively minimal

MMP in dim 2

アルゴリズム s.t. Input … proj. var X

Output … proj. var X'

with $\left\{\begin{array}{l} X' \text{は} X \text{と birational} \\ X' \text{は minimal (or Mori Fiber Space)} \end{array}\right.$ ←「ほぼ全単射」

Difficulty in dim ≥ 3

- X : smooth $\not\Rightarrow$ X' : smooth
- X : klt $\Longrightarrow$ X' : klt

でもそんなに悪くない特異点 (= klt) で閉じている。

§2. klt 特異点

$\boxed{\text{◎ canonical divisor}}$

① X : smooth var $/ k$.

$\rightsquigarrow \Omega_{X/k} = \{ \sum_i f\, dg \mid f_i, g_i \in \mathcal{O}_X \}$

$\rightsquigarrow \omega_X := \Omega_{X/k}^{\wedge \dim X} \; (:= \{ \sum f (dg_1 \wedge \cdots \wedge dg_n) \})$

- $K(X)$: function field

$\rightsquigarrow \Omega_{K(X)/k}^{\wedge \dim X} \ni \alpha \neq 0$

- 各 prime divisor $E \subset X$ について

$a_E :=$ "ω の $\underbrace{\eta_E}_{(E\text{ の generic pt})}$ における vanishing order" $\in \mathbb{Z}$

$\rightsquigarrow K_{X,\omega} := \sum_E a_E E \in \mathrm{Div}(X)$ を (ω の定める) canonical divisor

eg. 3

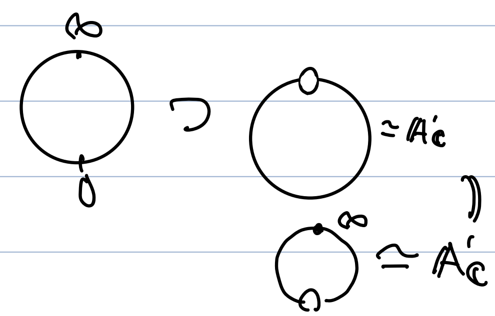
$X = \mathbb{P}^1_{\mathbb{C}}, \quad K(X) = \mathbb{C}(t)$

$\cup$

$U_1 = \mathrm{Spec}\, k[t]$

$U_2 = \mathrm{Spec}\, k[s] \quad (s = 1/t)$

$\Omega_{K(X)/\mathbb{C}} = K(X)\,dt \ni \omega := \frac{1}{t}\,dt$



U_1 において

- $\Omega_{U_1/\mathbb{C}} = \mathbb{C}[t]\,dt$ で

$\omega = 1/t\, dt$ とみる

ω は $\begin{cases} \text{零点} & \cdots \text{なし} \\ \text{pole} & \cdots \text{原点} \longrightarrow a_0 = -1 \end{cases}$

$K_{\mathbb{P}^1,\omega} = -0-\infty$

$\Omega_{U_2} = \mathbb{C}[s]\,ds \quad (s = 1/t)$

において $\omega = \frac{dt}{t} = s\,\frac{-ds}{s^2} = -\frac{1}{s}\,ds$

$\begin{cases} 0 \text{ 点なし} \\ \text{pole} \quad \{s=0\} = \infty, \quad a_\infty = -1 \end{cases}$

② $f: Y \to X$: birational w/ X, Y : smooth

$\Updownarrow$

$k(X) = k(Y)$

$$0 \neq \theta \in \Omega^{\wedge \dim X}_{k(X)/\mathbb{C}} = \Omega^{\dim Y}_{k(Y)/\mathbb{C}}$$

$$\rightsquigarrow \begin{cases} K_{X,\theta} \in \mathrm{Div}(X) \\ K_{Y,\theta} \in \mathrm{Div}(Y) \end{cases}$$

$$\rightsquigarrow K_{Y/X} := K_{Y,\theta} - f^* K_{X,\theta} \in \mathrm{Div}(Y)$$

θ に indep

Def

X : norm var / k : field

X : log terminal $\overset{def}{\Longleftrightarrow}$ ⓪ $K_{X,\theta}$: $\mathbb{Q}$-Cartier)

(= klt) ① $\forall f: Y \to X$: prop bir

(w/ Y : norm var)

($K_{Y/X}$ の全 ?a coeff) > -1.

Fact

X: norm & (K_X: $\mathbb{Q}$-Car)

$f: Y \to X$: log resol. $\Leftarrow$ (存在 $\Leftarrow$ 特異点解消 res. of sing.)

$\to X$: klt $\Leftrightarrow$ $K_{Y/X} > -1$

例えば… $\boxed{\dim \leq 3, \operatorname{char} 0 \Rightarrow \exists \log \text{ res.}}$

eg. $k = \mathbb{C}$, dim 2

(1) X: klt $\Leftrightarrow$ X: quotient sing.

i.e. $X \simeq \operatorname{Spec}(k[u,v]^G)$ $\exists G \subset GL_2(\mathbb{C})$: fin

更に X: klt & hyper sur. sing ($k[x,y,z]/(f)$)

$\Leftrightarrow$ X: RDP

rational double point

(Du Val, ADE… とも言われる)

$\Leftrightarrow$ $X \simeq k[u,v]^G$ ($G \subset SL_2$)

(3) $\operatorname{Spec}\left(k[x,y,z]/(x^a+y^b+z^c)\right)$: klt

$(2 \leq a \leq b \leq c)$

$\Leftrightarrow$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > 1$

$\iff (a,b,c) = \begin{cases} (2,2,*) \\ (2,3,3) \\ (2,3,4),\ (2,3,5) \end{cases}$

$\boxed{\text{⓪ 体と klt}}$

eg. k: 体, $R = k[x,y,z]/(z^2+x^3-ay^3+y^7)$

$\ni a$

	$\sqrt[3]{a} \notin k$	$\sqrt[3]{a} \in k$
$ch \neq 3$	③ klt	① klt
$ch = 3$	④ klt	② non-klt

① Hensel's lem $\xrightarrow[\begin{cases} ch \neq 3 \\ \sqrt[3]{a} \in k \end{cases}]{}$ $\underbrace{\sqrt[3]{a-y^4}}_{\ni u} \in k[[y]]$

$\rightsquigarrow R = k[x,y,z]/(z^2+x^3-y^3u^3)$

$\quad\quad \parallel \leftarrow$ 変数

$\quad\quad y^3$ 変換

② $(x - \sqrt[3]{a}\,y)^3 = x^3 \underbrace{- 3\sqrt[3]{a}\,x^2y + 3\sqrt[3]{a^2}\,xy^2}_{=0 \ \leftarrow \ ch = 3} - ay^3$

$= x^3 - ay^3$

→ ① $R \simeq \dfrac{k[\hat{x}, y, z]}{(z^2 + x^3 + yn)}$

③ $l := k(\sqrt[3]{a}) \to k \subset l$: sep'ble

$R \otimes_k l$: klt by ①

→ R : klt

⇐ lem

X : norm var / k, l/k : sep'ble

X : klt ⟺ $X \times_k l$, klt

" $\otimes_k l$ "

④ が一んばるとわかる

特に，③で使った lem が不成立 がわかる.

→ ⓒ $k(\sqrt[3]{a})/k$: insep.

→ ⓓ $\sqrt{Y}$: reg $\not\Rightarrow$ Y_e : not regular.

Part B Graph (cf. [Kollár '13] の和訳)

Aim … 2-dimensional klt を理解

§3 Dual graph

Setting ☆

以下, K : field, X : norm var / K of dim 2

$x \in X$: closed pt. w/ $\mathrm{Sing}(X) = \{x\}$.

$K(X)$: 商体

$\kappa(x) := \mathcal{O}_{X,x}/m_x$: 剰余体.

⓪ 交点数

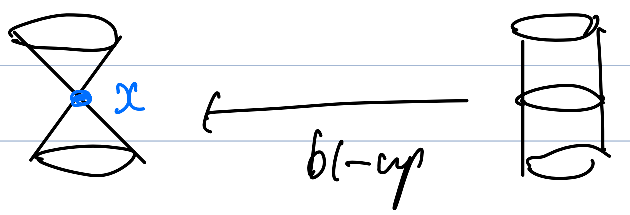
Fix $f: Y \to X$: resolution s.t.

$X \setminus \{x\}$ 上 isom.

Take $E \subset f^{-1}(x)$: irr compo.

e.g. $\mathrm{Spec}\left(\dfrac{k[x,y,z]}{(x^2+y^2-z^2)}\right)$

$Y = \mathbb{A}^1 \times \mathbb{P}^1$



$f^{-1}(x) = E \simeq \mathbb{P}^1_{\kappa(x)}$

$$\begin{tikzcd}
Y \arrow[r] & X \arrow[r] & \mathrm{Spec}(k) \\
f^{-1}(x) \arrow[u, hook] \arrow[r] & \mathrm{Spec}(\kappa(x)) \arrow[u, hook] & \\
E \arrow[u, phantom, "\cup" rotate=90] \arrow[ur] & &
\end{tikzcd}$$

① $P \subset T$ prime div. $P \neq E$

$\Longrightarrow P \cap E = \{p_1, \dots, p_e\}$

各 p_i の nbhd での P と E の def equation を それぞれ $f_i, g_i \in \mathcal{O}_{T,p_i}$

$\Longrightarrow m_i := \dim_{k(x)} \left(\mathcal{O}_{T,p_i} / (f_i, g_i) \right)$

$\rightsquigarrow (P, E) := \sum_i m_i$

Note : $A_i := \mathcal{O}_{T,p_i}/(f_i,g_i) \rightsquigarrow A_i$: Art local $/ k(x)$

res $\cong k(p_i)$

$\rightsquigarrow m_i = \dim_{k(x)}(A_i)$

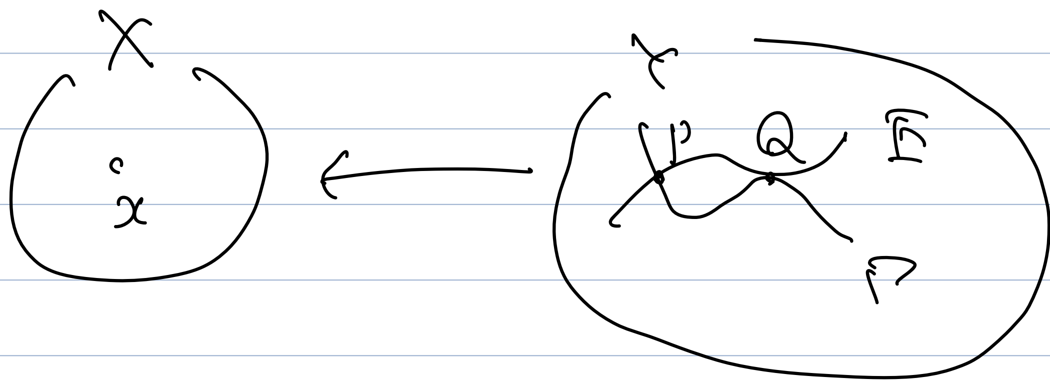
$= \ell(A_i) \cdot [k(p_i) : k(x)]$

$\|$

$\dim_{k(p_i)}(A_i)$

[illegible]

eg.（お気持ち）



$K(P) = \mathbb{C}$ ⇝ $m_P = 1\cdot 2 = 2$

$K(Q) = \mathbb{R}$ $\quad m_Q = 2\cdot 1 = 2$

$(P\cdot E) = 2+2 = 4$

② 自己交点数 $(E\cdot E) = (E^2) \in \mathbb{Z}_{<0}$

がうまく定まる（略）

$$\leadsto \begin{cases} \mathrm{Div}(Y) \to \mathbb{Z}\ ;\ D \mapsto (D\cdot E) \\ \mathrm{Div}(Y) \to \mathbb{Q} \end{cases}$$

$\boxed{\text{③ min. res}}$

Def

- X as in Setting ⊛
- min. I res of X is $f: Y \to X$: res s.t. $\forall g: Z \to X$: res

$$\begin{tikzcd} Z \arrow[r, "\exists"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \\ & X \supset U \end{tikzcd} \qquad \begin{tikzcd} g^{-1}(U) \arrow[r] \arrow[dr] & f^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

Fact $\exists 1$ min'l res

$\boxed{\text{◎ 諸性質}}$

$x \in X$ as in ⊛

$f: Y \to X$: minimal res

$\rightsquigarrow f^{-1}(x) = \bigcup_i E_i$: irre decomp.

$$\begin{cases} r_i := \dim_{k(x)}(H^0(\mathcal{O}_{E_i})) \in \mathbb{Z}_{>0} \\ g_i := \dim_{k(x)}(H^1(\mathcal{O}_{E_i})) \end{cases}$$

$$\frac{}{r_i} \in \mathbb{Z}_{\geq 0}$$

種数

eg: $k(x) = \mathbb{R}$, $E_i \simeq \mathbb{P}^1_{\mathbb{C}} \rightsquigarrow \begin{cases} H^0(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = \mathbb{C} \to r_i = 2 \\ g_i(\mathbb{P}^1_{\mathbb{C}}) = 0 \end{cases}$

Rem: $x \in X$: klt ならば (実は) 全ての E_i は

$\mathbb{P}^1_{\mathbb{Q}}$ の form たちのもの

特に r_i は $= 1, 2$ の情報を記憶しているだけ

Fact

(1) (cnt'dness) $\bigcup_i E_i$: cnt'ed

(2) (divisible) $\forall i$, $\forall D \in \mathrm{Div}(Y)$, $r_i \mid (D, E_i)$

(3) (Hodge index thm) $\forall D = \sum_i a_i E_i \neq 0$ $(a_i \in \mathbb{Q})$

$$\longrightarrow (D^2) < 0$$

$$(D^2) = (D, D) = \sum a_i a_j (E_i, E_j)$$

$i, j.$

(4) (f: minimal) $(K_{Y/X} \cdot E_i) \geq 0$ $(\forall i)$

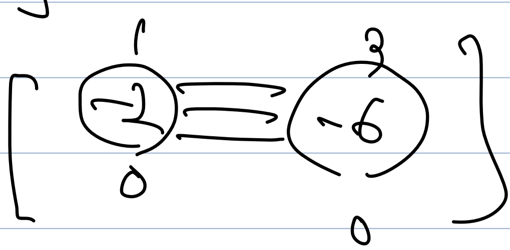
(5) (Riemann-Roch
+ adjunction)
$(K_{Y/X}, E_i) = -(E_i^2) + 2 r_i (g_i - 1)$

(6) (vanishing) $\exists i,\ r_i = 1$

$\boxed{\text{① Dual graph}}$

Today … graph = $\mathbb{Z}^3$の重みつき無向多重グラフ

e.g.



頂点…2つ

重み…それぞれ $\begin{pmatrix}1\\-2\\0\end{pmatrix}, \begin{pmatrix}3\\-6\\0\end{pmatrix} \in \mathbb{Z}^3$

辺…3本

Def. $(x \in X)$ as in Setting ⊛

$\to$ dual graph $\Gamma(x \in X)$ を次で def:

① $f : Y \to X$: min. res

$\to$ 頂点 … 各 E_i

重み … $\begin{pmatrix} r_i \\ b_i \\ g_i \end{pmatrix} \in \mathbb{Z}^3$

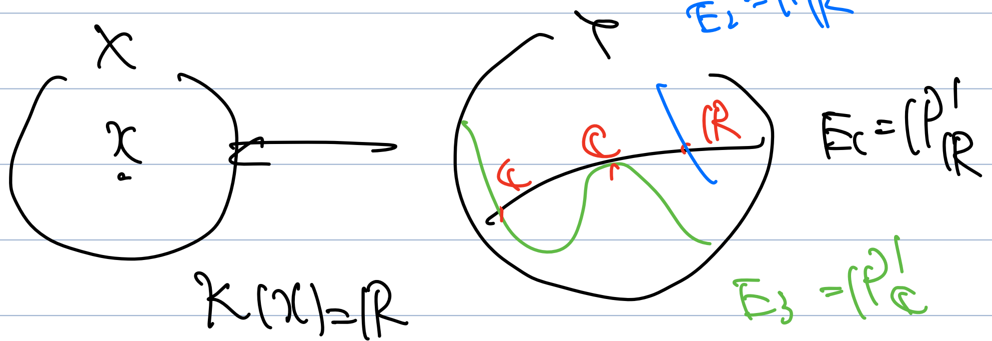
$$f^{-1}(x) = \bigcup_i E_i$$

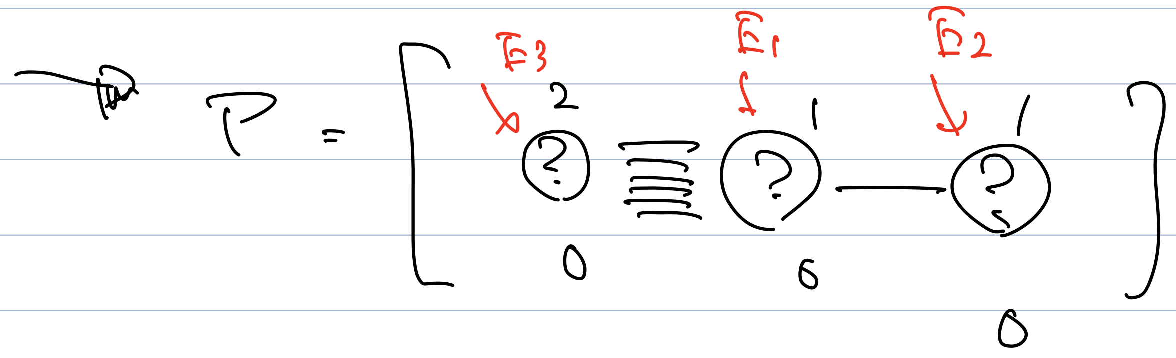
where $r_i \in \mathbb{N}_{\geq 1}$, $g_i \in \mathbb{N}$ はさっきと同じ

$b_i := (E_i^2) \in \mathbb{Z}_{<0}$

・辺 … E_i と E_j の間に $(E_i \cdot E_j)$ 本.

eg. (お気持ち)





§5 分類

Prop

G: Graph

G: klt $\iff$ G は次のどれか:

(I) (Chain) $\left[\underset{0}{\overset{1}{(b_1)}} - \underset{0}{\overset{1}{(b_2)}} - \cdots - \underset{0}{\overset{1}{(b_n)}} \right]$ (w/ $b_i \leq -2$)

以下, $*$ = -2以下なる何でもOK,

○の上に何も書かない $\rightarrow$ $r_i = 1$

(下) $\rightarrow$ $g_i = 0$ とする.

(つまり (I) は $(*) - (*) - \cdots - (*)$ となる)

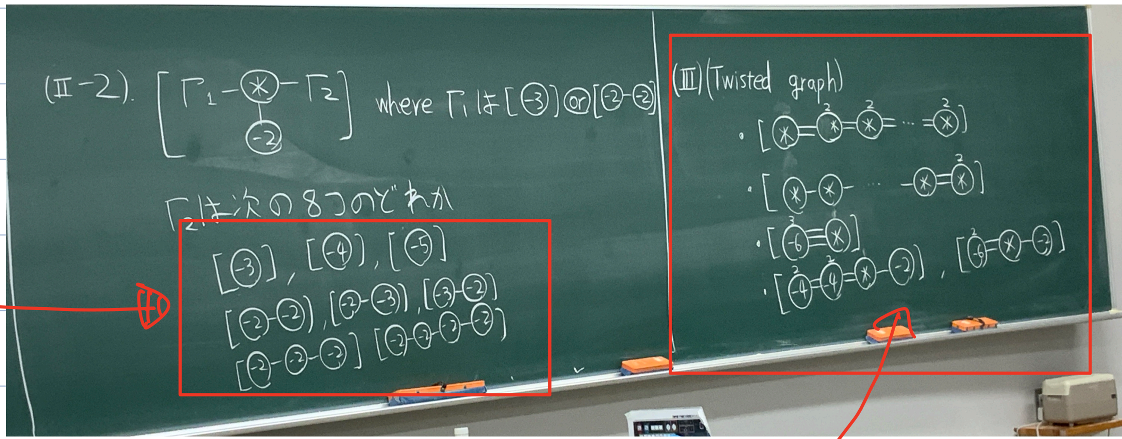
(II) (One-fold)

(II-1) $\left((-2) - \underset{\substack{| \\ (-2)}}{(*)} - (*) - \cdots - (*) \right)$

(II-2) $\left[\Gamma_1 - \underset{\substack{| \\ (-2)}}{(*)} - \Gamma_2 \right]$ where Γ_1 は $[(-3)]$ or

$[(-2) - (-2)]$

Γ_2は次の8つのうちのどれか.



(III) (Twisted graph)