2025.6.7 第9回 可換代数と組合せ論セミナー(東邦大学)
グラフ超平面配置とその変形(辻栄周平 北海道教育大学旭川校)

§自由配置

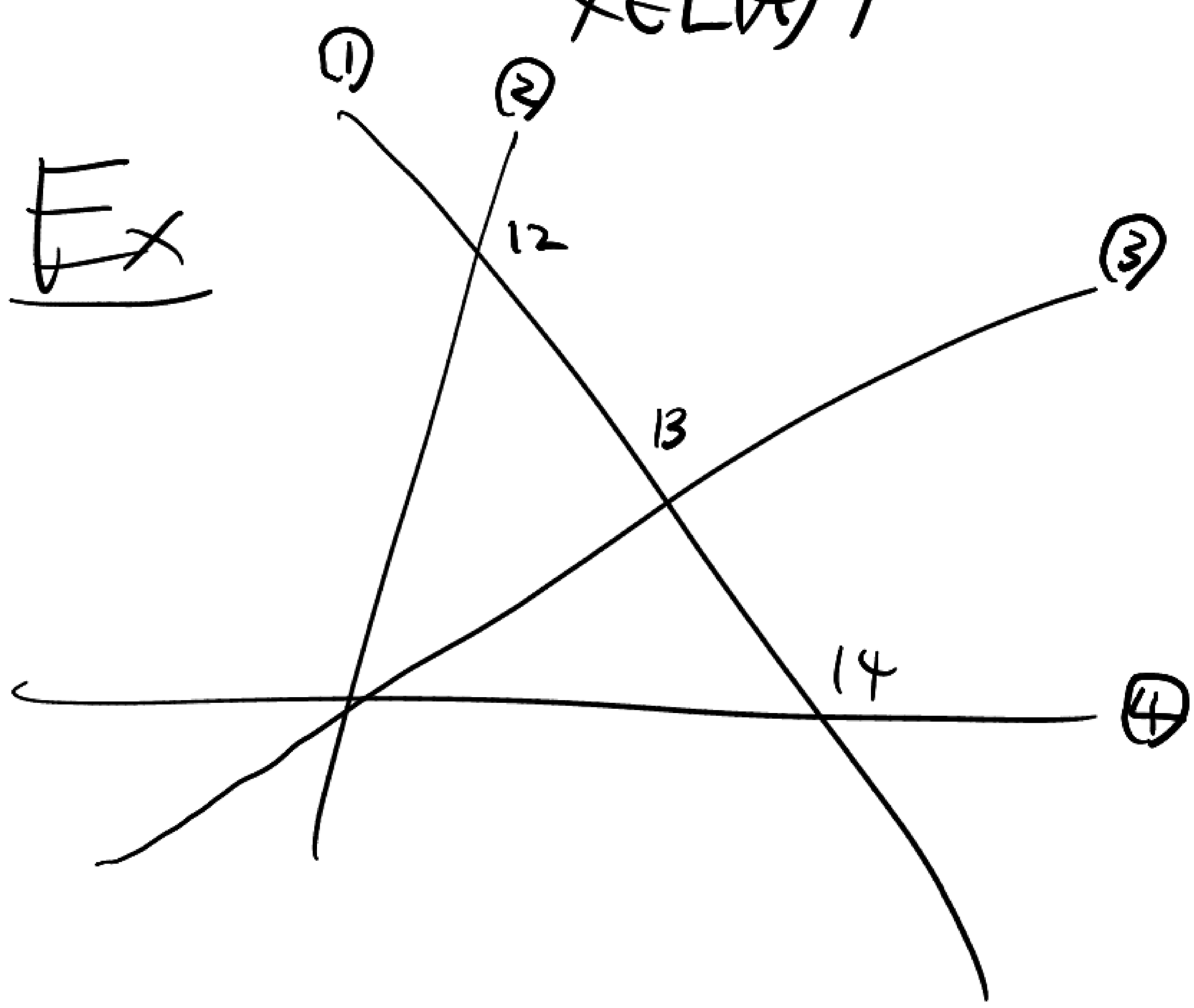
$\mathcal{A} = \{H_1, \dots, H_n\}$ $\quad H_i \subseteq \mathbb{K}^\ell$: hyperplane

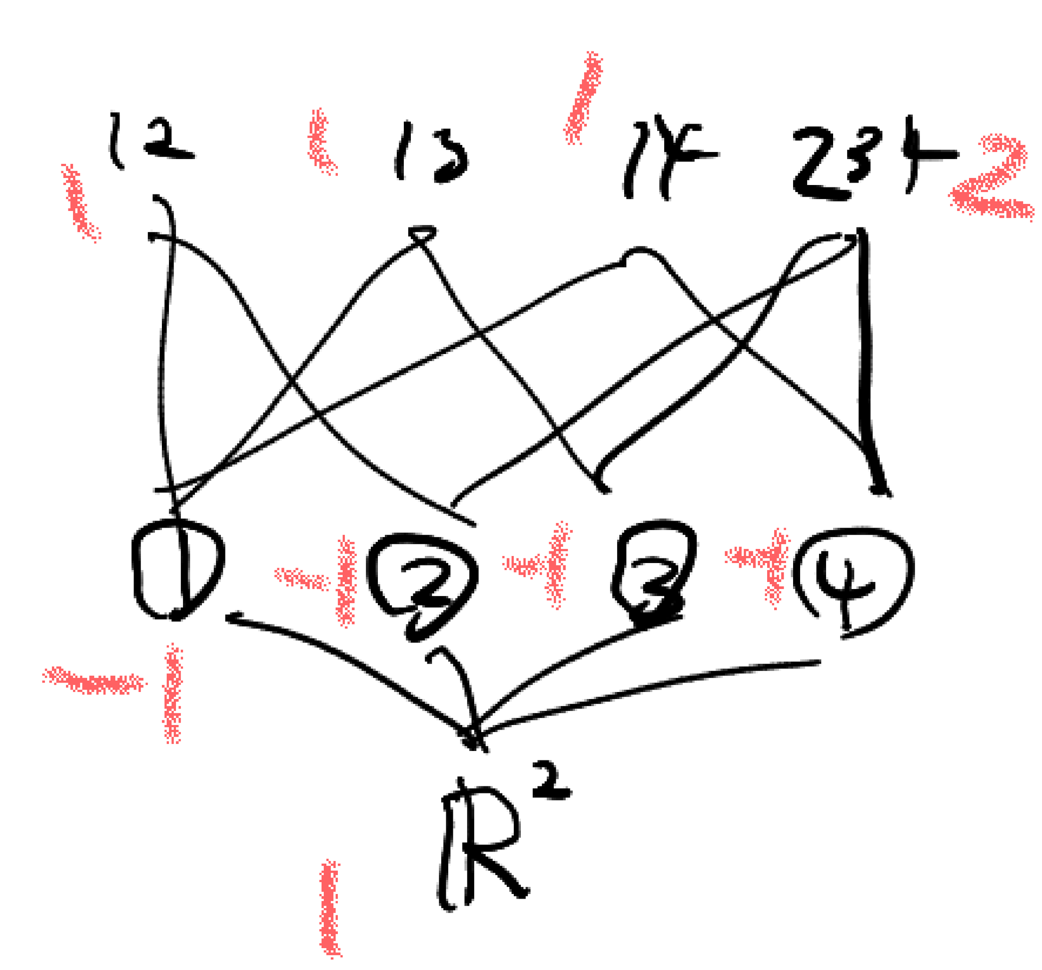
$L(\mathcal{A}) := \{ \bigcap_{H \in \mathcal{B}} H \neq \emptyset \mid \mathcal{B} \subseteq \mathcal{A} \}$: intersection poset

$$X \leq Y \overset{\mathrm{def}}{\iff} X \supseteq Y$$

$$\chi_{\mathcal{A}}(t) := \sum_{X \in L(\mathcal{A})} \mu(X)\, t^{\dim X}$$

Ex





$$\chi_{\mathcal{A}}(t) = t^2 - 4t + 5$$

$S := \mathbb{K}[x_1, \cdots, x_\ell]$

$\mathrm{Der}_S := \left\{ \theta : S \to S \;\middle|\; \begin{array}{l} \theta \text{ は } \mathbb{K}\text{-linear} \\ \theta(fg) = \theta(f)g + f\theta(g) \end{array} \right\}$

$\simeq \bigoplus_{i=1}^{\ell} S\partial_i \quad , \quad \partial_i := \dfrac{\partial}{\partial x_i}$

$\mathcal{A}$: central i.e. $H \ni 0$ for ${}^\forall H \in \mathcal{A}$.

$H =: \ker \alpha_H$

$D(\mathcal{A}) := \{ \theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H \text{ for } {}^\forall H \in \mathcal{A} \}$

Def $\mathcal{A}$: free $\overset{\text{def}}{\Longleftrightarrow}$ $D(\mathcal{A})$: free S-module.

$\{\theta_1, \cdots, \theta_\ell\}$ が $D(\mathcal{A})$ の斉次基底であるとき,

$\exp(\mathcal{A}) := (\deg\theta_1, \cdots, \deg\theta_\ell)$

Thm (Saito's criterion)

$\theta_1, \cdots, \theta_\ell \in D(\mathcal{A})$: homogeneous. TFAE

(1) $\{\theta_1, \cdots, \theta_\ell\}$ は $D(\mathcal{A})$ の基底

(2) $\{\theta_1, \cdots, \theta_\ell\}$ は S 上1次独立 かつ $\sum_{i=1}^{\ell} \deg\theta_i = |\mathcal{A}|$.

(3) $\det M(\theta_1, \cdots, \theta_\ell) \doteq Q$

ただし，

$$M(\theta_1, \cdots, \theta_\ell) := (\theta_j(x_i))$$

$$Q_{\mathcal{A}} := \prod_{H \in \mathcal{A}} \alpha_H$$

<u>Ex.</u> $Q_{\mathcal{A}} = x_1 x_2 (x_1 + x_2)$

$\theta_1 = x_1 \partial_1 + x_2 \partial_2$

$\theta_2 = x_1^2 \partial_1 - x_2^2 \partial_2$

$\theta_1, \theta_2 \in D(\mathcal{A})$

$$\det M = \begin{vmatrix} x_1 & x_2 \\ x_1^2 & -x_2^2 \end{vmatrix} = -x_1 x_2^2 - x_1^2 x_2 = -x_1 x_2 (x_1 + x_2) \doteq Q$$

$\therefore$ $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, 2)$

<u>Ex</u> $Q_{\mathcal{A}} = x_1 \cdots x_\ell$: Boolean arr.

$\theta_i = x_i \partial_i \in D(\mathcal{A})$ $\quad \det M = \begin{vmatrix} x_1 & & \\ & \ddots & \\ & & x_\ell \end{vmatrix} = Q$

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, \cdots, 1)$

<u>Thm</u> (Terao's factorization theorem)

$\mathcal{A}$: free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$.

Then $\chi_{\mathcal{A}}(t) = (t-d_1)\cdots(t-d_\ell)$

<u>Rem</u> $\chi_{\mathcal{A}}(t)$ が分解するが、自由でないことがある.

<u>Conj</u> (Terao)

$\mathcal{A}, \mathcal{B} / \mathbb{K}$. $L(\mathcal{A}) \simeq L(\mathcal{B})$.

$\mathcal{A}$ is free $\Longleftrightarrow$ $\mathcal{B}$ is free

<u>Rem</u> $\mathcal{A}$ と $\mathcal{B}$ の定義体の標数が異なるときは、

$L(\mathcal{A}) \simeq L(\mathcal{B})$, $\mathcal{A}$ は自由 $\mathcal{B}$ は自由でない.

という例がある.

Ex $Q_{\mathcal{A}} = \prod_{1 \le i < j \le \ell} (x_i - x_j)$ braid arr.

$$\theta_i := x_1^{i-1} \partial_1 + \cdots + x_\ell^{i-1} \partial_\ell \in D(\mathcal{A})$$

$$\det M = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_\ell \\ \vdots & \vdots & & \vdots \\ x_1^{\ell-1} & x_2^{\ell-1} & \cdots & x_\ell^{\ell-1} \end{vmatrix} \doteq Q_{\mathcal{A}}$$

Vandermonde determinant

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (0, 1, 2, \cdots, \ell-1)$

$$\chi_{\mathcal{A}}(t) = t(t-1)\cdots(t-\ell+1)$$

$$L(\mathcal{A}) \simeq \{ \text{set partitions of } [\ell] \}$$

$$\bigcap_{k=1}^{r} \bigcap_{i,j \in B_k} \{x_i = x_j\} \longleftarrow \pi = \{B_1, \cdots, B_r\}$$

Ex

$$\{x_1 = x_2 = x_3\} \cap \{x_4 = x_5\} \longleftrightarrow 123|45$$

Rem

複素鏡映配置も
自由.

Ex (Dedekindの定理)

G: finite abelian group

$(x_g)_{g\in G}$

$$\Delta(G) := \det\left(x_{g^{-1}h}\right)_{g,h\in G} = \prod_{\chi\in \mathrm{Hom}(G,\mathbb{C}^\times)} \sum_{g\in G} \chi(g) x_g$$

$$\Delta(\mathbb{Z}/2\mathbb{Z}) = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_0 \end{vmatrix} = x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1)$$

$$\Delta(\mathbb{Z}/3\mathbb{Z}) = \begin{vmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{vmatrix} = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2$$

$$= (x_0 + x_1 + x_2)(x_0 + \omega x_1 + \omega^2 x_2)(x_0 + \omega^2 x_1 + \omega x_2).$$

∵ $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ に対し、 $a_\chi := \sum_{g\in G} \chi(g) x_g$

指標の1次独立性より、 $\{a_\chi\}_\chi$ は $\mathbb{C}$ 上1次独立.

$\mathcal{A} := \{\ker a_\chi\}_\chi$ は Boolean arr と線形同値.

∴ free with $\exp(\mathcal{A}) = (1, 1, \cdots, 1)$.

$\theta_g := \sum_{h\in G} x_{g^{-1}h}\partial_h$ とすると、

$$\theta_h(\alpha_\chi) = \left(\sum_{h\in G} x_{g^{-1}h}\partial_h\right)\left(\sum_{h\in G}\chi(h)x_h\right)$$

$$= \sum_{h\in G}\chi(h)x_{g^{-1}h}$$

$$= \sum_{k\in G}\chi(gk)x_k$$

$$= \chi(g)\sum_{k\in G}\chi(k)x_k$$

$\therefore\ \theta_h \in D(A)$.

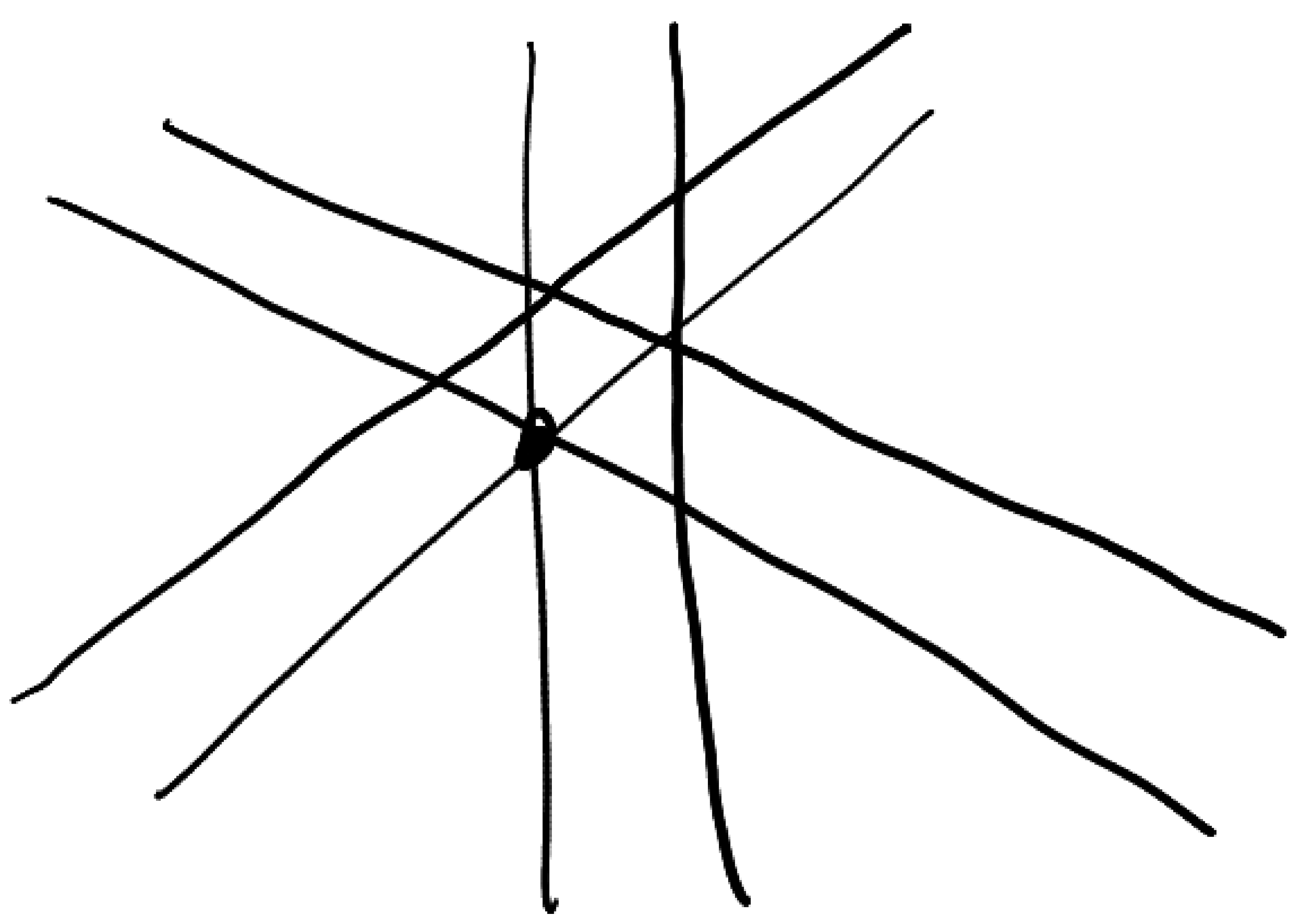
$\det M$ の x_0^{ℓ} の係数は 1 なので $\det M \neq 0$.

よって、$\{\theta_h\}$ は $D(A)$ の基底.

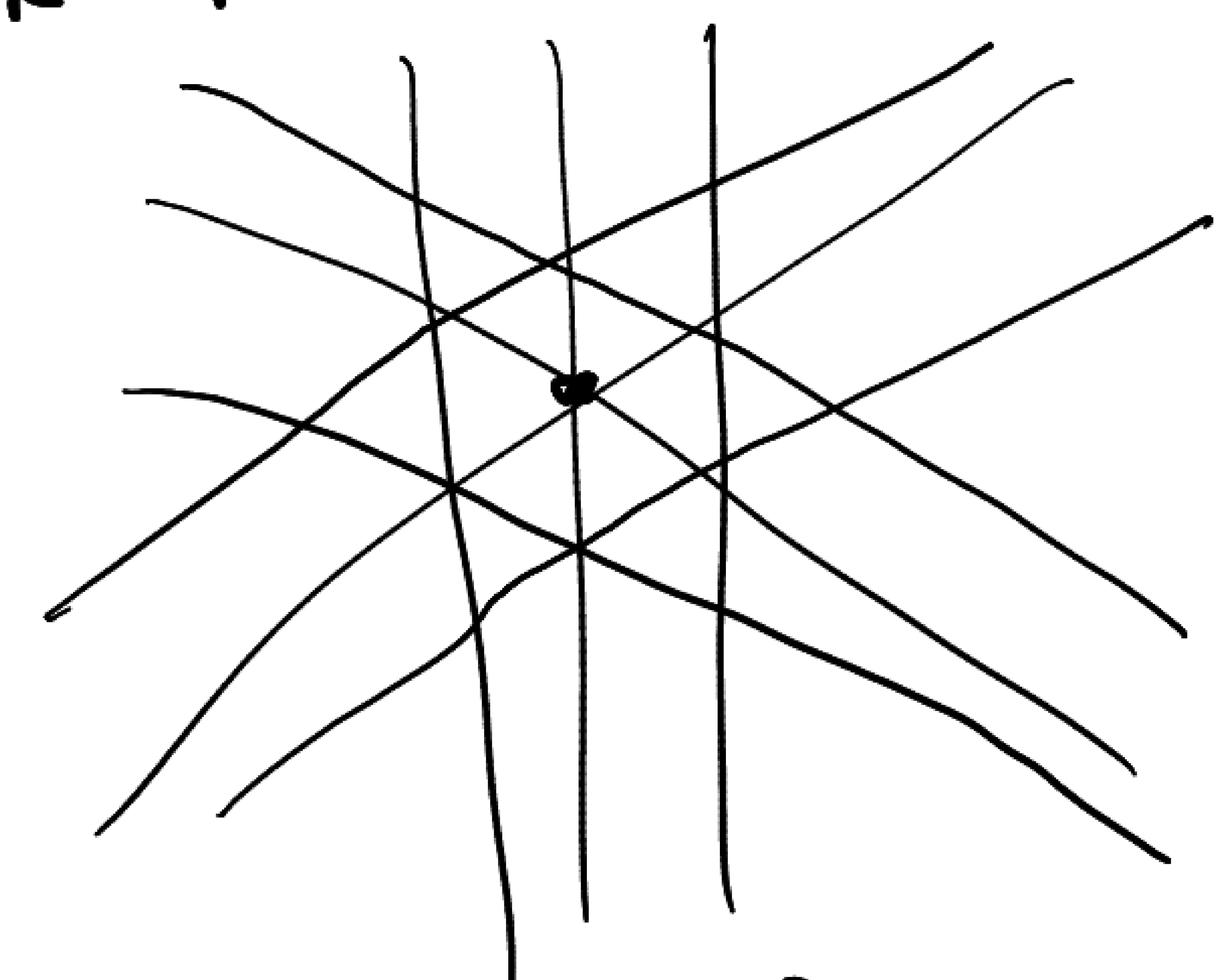
$\therefore\ \det M = Q_A = \sum_{g\in G} \alpha_g$.

Ex $\mathrm{Shi}(l,m)$: $\displaystyle\prod_{\substack{1\le i<j\le l \\ 1-m\le k\le m}} (x_i - x_j - k)$

$\mathrm{Cat}(l,m)$: $\displaystyle\prod_{\substack{1\le i<j\le l \\ -m\le k\le m}} (x_i - x_j - k)$



$\mathrm{Shi}(2,1)$



$\mathrm{Cat}(2,1)$

$\mathrm{Shi}(l,m)$ と $\mathrm{Cat}(l,m)$ の cone は自由.

~~基底構成~~ : Suyama-Yoshinaga (~~和分表示~~)

flatの数え上げ : Nakashima-T (combinatorial species)

Rem 他のルート系でも, Shi, Cat は定義でき

自由であることが知られている (Yoshinaga)

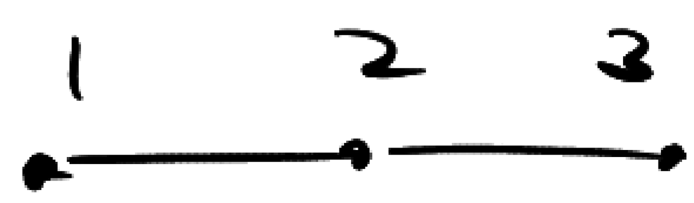
B型Catalanの基底構成, Kawanoue

§ グラフ配置

$G = ([\ell], E)$: simple graph

$$Q_{\mathcal{A}_G} := \prod_{ij \in E} (x_i - x_j) \text{ : グラフ配置}$$

Ex.



$Q_{\mathcal{A}_G} = (x_1 - x_2)(x_1 - x_3)$

Ex. $\mathcal{A}_{K_\ell}$: Braid arr.

Thm $\chi_{\mathcal{A}_G}(t) = \chi_G(t)$

Thm

(1) G は PEO をもつ

(2) $\mathcal{A}_G$ は supersolvable

(3) $\mathcal{A}_G$ は free

(4) G は chordal

$\mathcal{A}_G$]

MAT-free

$\cap\mid$

free

$\cap\mid$

$\operatorname{pdim} D(\mathcal{A}_G) = 1$

$\cap\mid$

$\operatorname{pdim} D(\mathcal{A}_G) = 2$

G]

strongly chordal (Tran-T)

$\cap\mid$

chordal

$\cap\mid$

weakly chordal (cite)

$\cap\mid$

?

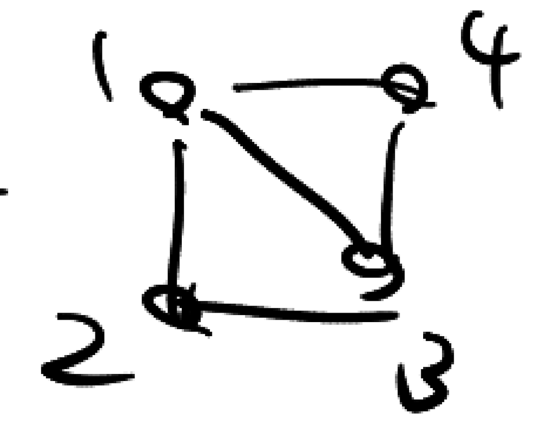
基底構成 : Suyama-T

$(v_1, \dots, v_\ell)$: PEO

$$C_{\ge k} := \{k\} \cup \left\{ i \in [\ell] \;\middle|\; \begin{array}{l} \exists\, \text{path } v_k v_{j_1} \cdots v_{j_n} v_i \\ \text{s.t. } k < j_1 < \cdots < j_n < i \end{array} \right\}$$

$$E_{<k} := \left\{ j \in [\ell] \;\middle|\; j < k,\ \{v_j, v_k\} \in E \right\}$$

$$\theta_k := \sum_{i \in C_{\ge k}} \frac{\Delta(E_{<k}, x_i)}{\Delta(E_{<k})} \partial_i$$

Ex $G =$  $\exp(A_G) = (0, 1, 2, 2)$

$C_{\geq 1} = \{1,2,3,4\}$. $E_{<1} = \emptyset$. $\theta_1 = \partial_1 + \partial_2 + \partial_3 + \partial_4$

$C_{\geq 2} = \{2,3,4\}$ $E_{<2} = \{1\}$ $\theta_2 = (x_2 - x_1)\partial_2 + (x_3 - x_1)\partial_3 + (x_4 - x_1)\partial_4$

$C_{\geq 3} = \{3,4\}$. $E_{<3} = \{1,2\}$. $\theta_3 = (x_3 - x_1)(x_3 - x_2)\partial_3 + (x_4 - x_1)(x_4 - x_2)\partial_4$

$C_{\geq 4} = \{4\}$ $E_{<4} = \{1,3\}$. $\theta_4 = (x_4 - x_1)(x_4 - x_3)\partial_4$

Problem $G \supseteq K_m$ のとき.

$$\chi_{K_m}(t) = t(t-1)\cdots(t-m+1) \mid \chi_G(t).$$

G が chordal のとき.

$$\exp(K_m) = (0, 1, \cdots, m-1) \subseteq \exp(A_G)$$

$D(A_{K_m})$ の基底を $D(A_G)$ の基底に延長できるか?

$D(A_{K_3})$ の基底 $\theta_1 = \partial_1 + \partial_2 + \partial_3$

$\theta_2 = x_1\partial_1 + x_2\partial_2 + x_3\partial_3$

$\theta_3 = x_1^2\partial_1 + x_2^2\partial_2 + x_3^2\partial_3$

$\psi_1 = \theta_1 + \partial_4$

$\psi_2 = \theta_2 + x_4\partial_4$ $\psi_4 = (x_4 - x_1)(x_4 - x_3)\partial_4$

$\psi_3 = \theta_3 + x_3^2\partial_4$

$$\det M = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ 0 & 0 & 0 & (x_4-x_1)(x_4-x_3) \end{vmatrix} \doteq Q_{A_G}.$$

Chordalグラフはクリークを貼りあわせてできているので、貼りあわせの構造を反映した基底が作れるかもしれない.

<u>Thm</u> (T) 以下の操作で生成されるクラス $\mathcal{C}$ に含まれる arr は free

(1) $\emptyset \in \mathcal{C}$

(2) $A \in \mathcal{C}$, A は B の modular coatom $\Rightarrow B \in \mathcal{C}$

(3) $A_1, A_2 \in \mathcal{C} \Rightarrow A_1$ と A_2 の modular round flat での貼り合わせも $\mathcal{C}$ に属する.

<u>Problem</u> より一般に、2つの free arr. の modular flat での はりあわせも自由か?

§ グラフ多重配置

$$\mu : \mathcal{A} \longrightarrow \mathbb{Z}_{\geq 0}$$

$(\mathcal{A}, \mu)$ を多重配置という.

$$D(\mathcal{A}, \mu) = \{ \theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^{\mu(H)} \text{ for } \forall H \in \mathcal{A} \}$$

<u>Def</u> $(\mathcal{A}, \mu)$: free $\Longleftrightarrow$ $D(\mathcal{A}, \mu)$: free as S-mod.

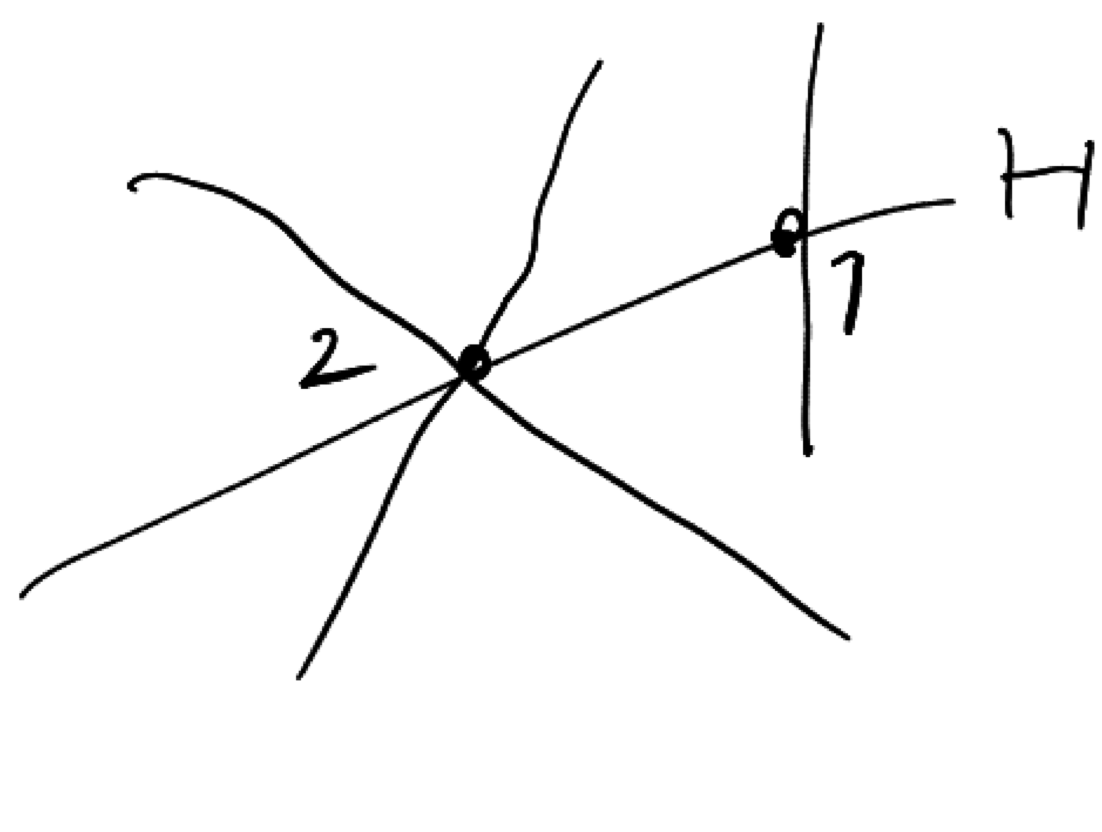
$$\mathcal{A}^H := \{ H \cap K \mid K \in \mathcal{A} \setminus \{H\} \}$$

$\mathcal{A}$ が free でも $\mathcal{A}^H$ が free とは限らない.

<u>Thm</u> (Ziegler) $H \in \mathcal{A}$.

$\mathcal{A}$: free $\Longrightarrow$ $(\mathcal{A}^H, m_H)$: free

$$m_H(X) := \# \{ K \in \mathcal{A} \setminus \{H\} \mid K \supset X \}$$



$X \in L(\mathcal{A})$. $\mathcal{A}_X := \{ H \in \mathcal{A} \mid H \supseteq X \}$.

$\mathcal{A}$ が free なら $\mathcal{A}_X$ も free.

とくに X が modular なら $\exp(\mathcal{A}_X^{\mathrm{ess}}) \subseteq \exp(\mathcal{A})$.

Thm (Yoshinaga) $\ell \geq 4$. $H_0 \in \mathcal{A}$. TFAE.

(1) $\mathcal{A}$: free.

(2) $(\mathcal{A}^{H_0}, m_{H_0})$ is free and $\mathcal{A}_X$ is free for $\forall X \in L(\mathcal{A}^{H_0}) \setminus \{\hat{1}\}$

多重配置は重要だが、braid arr の場合でも

よくわからない.

- constant multiplicity なら free

  ど Shi, Cat の Ziegler 制限.

- Abe-Naida-Numata.

  const ± 1 の freeness の特徴づけ.

- Wakamiko $\ell = 3$ (常に自由) の　　$\boxed{x^a y^b (x+y)^c}$

  exp の決定.

  基底の構成 (一般二項係数)

- T-Uchiumi, Wakamiko $x^a y^b (x+y)^c$ の正標数版
    - exp を実効的に求められるアルゴリズム
    - 基底はよく分かっていない

- Bandlow-Musiker, Feigin

odd const mult の場合の基底

$$\theta_k = \sum_{i,j=1}^{l} \left(\int_{x_i}^{x_j} t^k \left(\prod_{p=1}^{l} (t - x_p) \right)^m dt \right) \partial_i$$

- Suyama-Toshinaga

Cat の基底（離散化する）

$$\theta_k = \sum_{i,j=1}^{l} \left(\sum\nolimits_{x_i}^{x_j} t^k \left(\prod_{p=1}^{l} (t - x_p) \right)^{\underline{m}} \Delta t \right) \partial_i$$

$\Delta f(t) := f(t+1) - f(t)$ 差分

$\sum \Delta t$：和分（差分の逆操作）

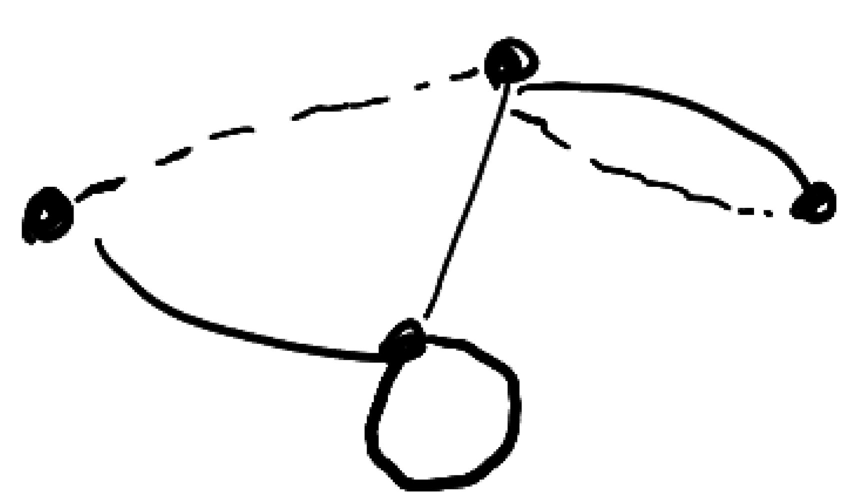
$f(t)^{\underline{m}} := f(t) f(t-1) \cdots f(t-m+1)$

Problem

- braid multiarr (グラフ multiarr) の自由性の特徴づけ.
- 基底の構成

§ Signed graphic arr.

$$G = ([\ell], E^+, E^-, L)$$



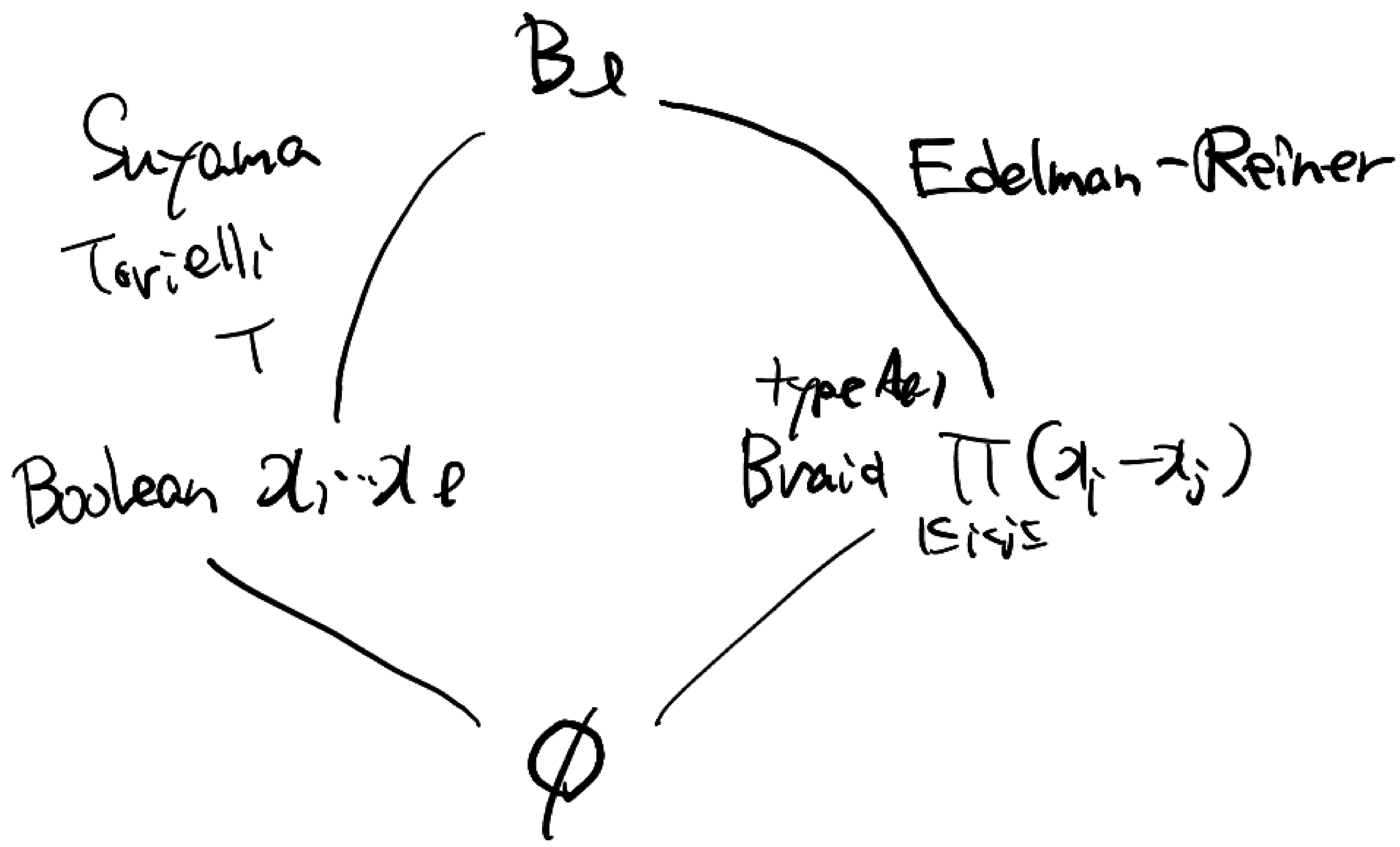
$ij \in E^+ \Leftrightarrow x_i - x_j = 0$

$ij \in E^- \Leftrightarrow x_i + x_j = 0$

$i \in L \Leftrightarrow x_i = 0$

Type B Coxeter arr. $Q = x_1 \cdots x_\ell \prod_{1 \le i < j \le \ell} (x_i^2 - x_j^2)$

の subarr をすべて表せる.



Problem

- complete characterization
- 基底の構成
- $x_1\cdots x_\ell \prod_{1\le i<j\le \ell}(x_i^n - x_j^n)$ について同様の問題

§ Integral gain graphic arr.

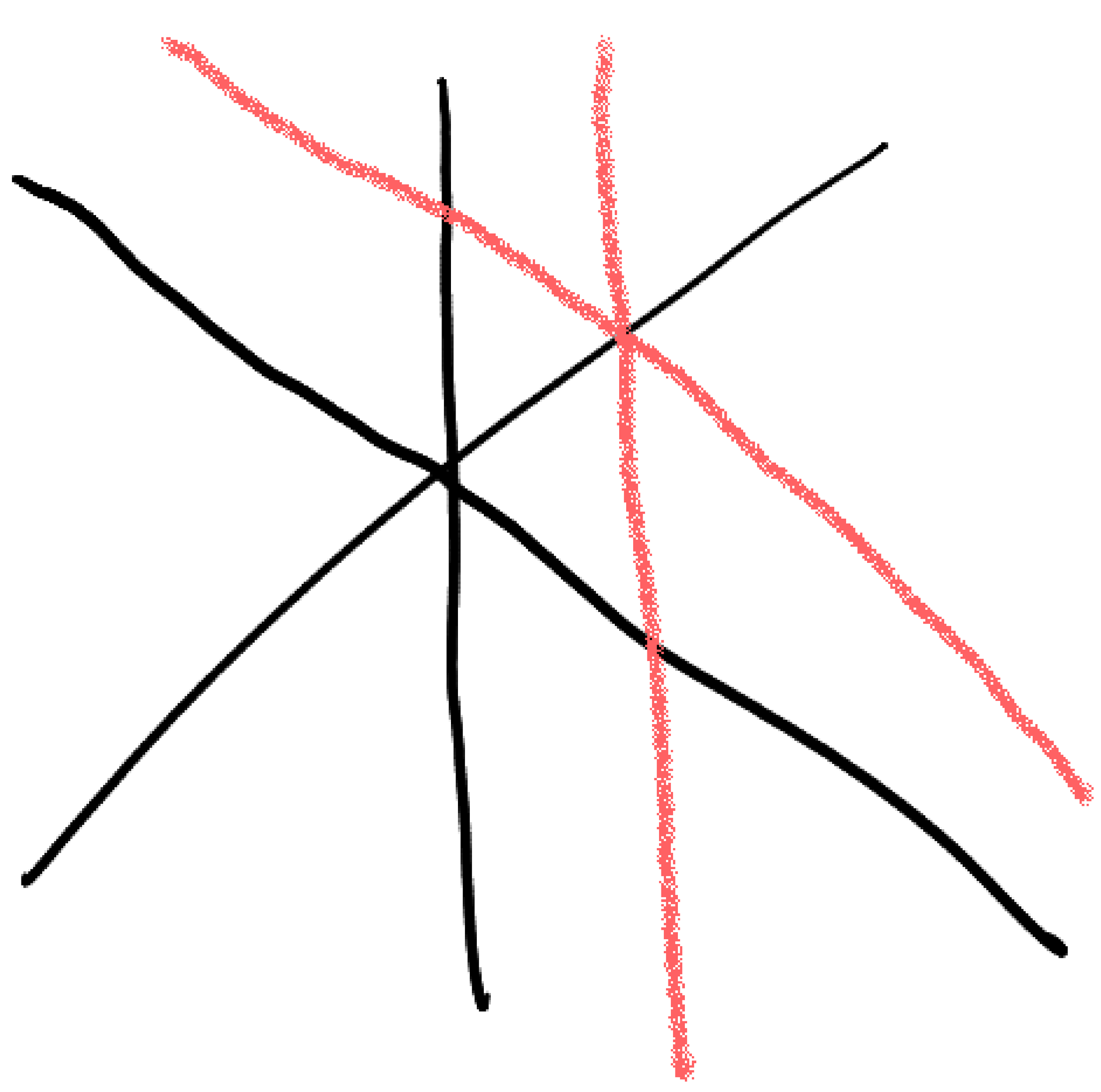
D : acyclic digraph on $[\ell]$, $\gamma \in \mathbb{C}^\times \setminus \{1のべき根\}$

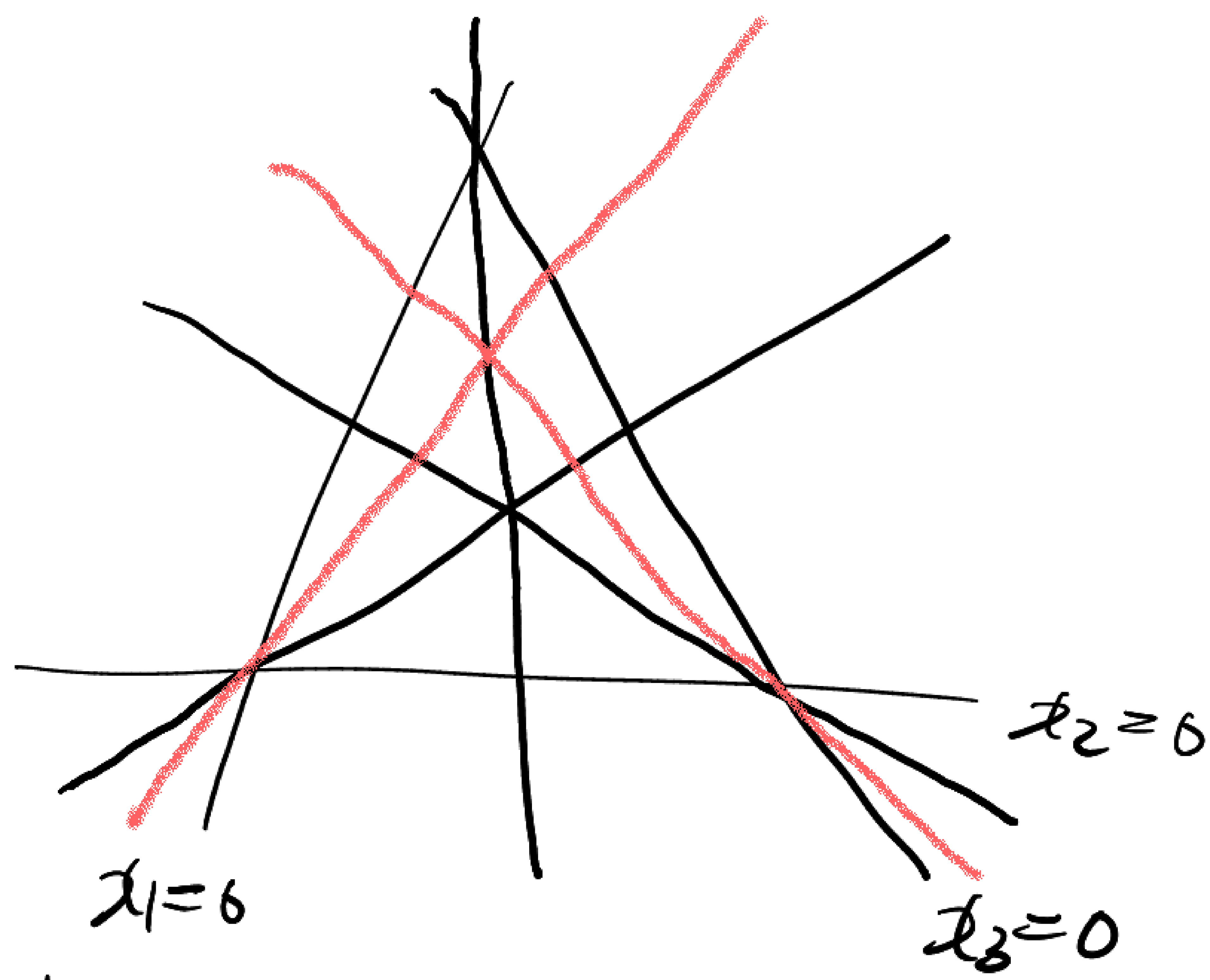
$$\mathcal{A}(D) : \prod_{1 \le i < j \le \ell} (x_i - x_j) \prod_{(i,j) \in E} (x_i - x_j - 1)$$

$$\mathcal{B}(D) : x_1 \cdots x_\ell \prod_{1 \le i < j \le \ell} (x_i - x_j) \prod_{(i,j) \in E} (x_i - \gamma x_j)$$

<u>Ex.</u> $D =$

$$\begin{tikzcd} & & 1 \arrow[dll] \\ 2 \arrow[rr] & & 3 \end{tikzcd}$$





<u>Thm</u> (Athanasiadis, Bailey) TFAE

(1) $c\mathcal{A}(D)$ is free

(2) $\mathcal{B}(D)$ is free

(3) D is $\left(\begin{tikzcd}[sep=small] & \circ \arrow[dl] \\ \circ \arrow[r] & \circ \end{tikzcd},\ \begin{tikzcd}[sep=small] \circ \arrow[d] & \circ \arrow[d] \\ \circ & \circ \end{tikzcd} \right)$-free.

各辺に整数ラベルをつけた有向グラフを考える.

$i \xrightarrow{k} j$ 同一視 $i \xleftarrow{k} j$

$\{x_i - x_j = k\} \;=\; \{x_j - x_i = -k\}$

$\{x_i - \varepsilon^k x_j = 0\} \;=\; \{x_j - \varepsilon^{-k} x_j = 0\}$

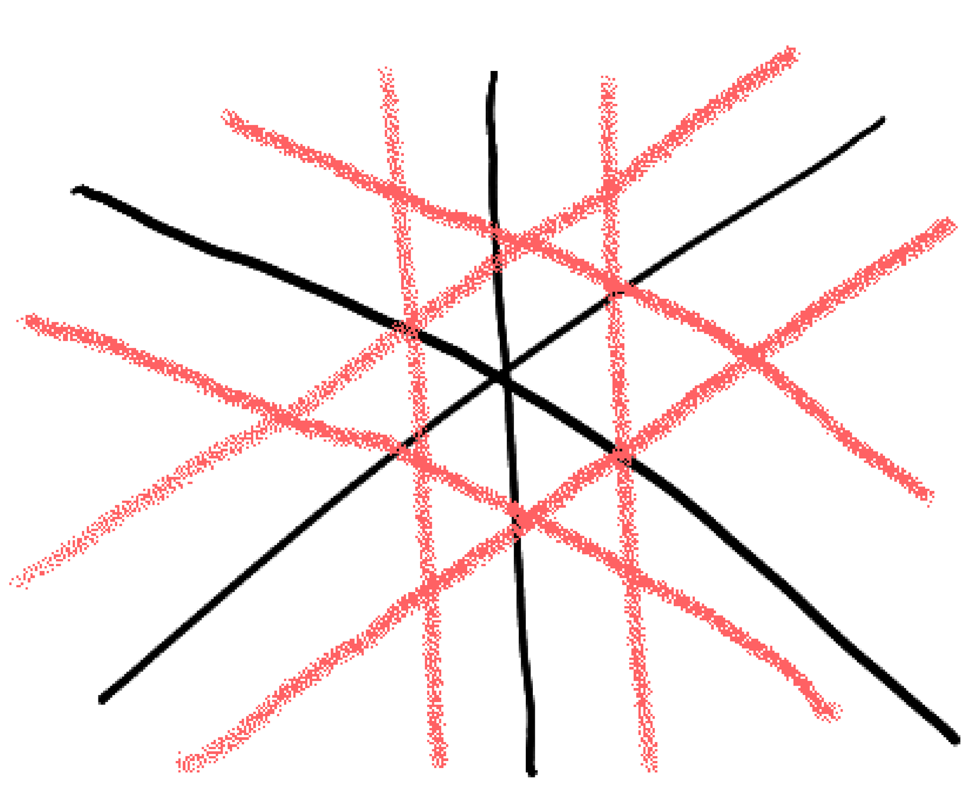
$\Gamma = ([\ell], E)$

$A(\Gamma) : \prod_{[i,j;k]\in E} (x_i - x_j - k)$

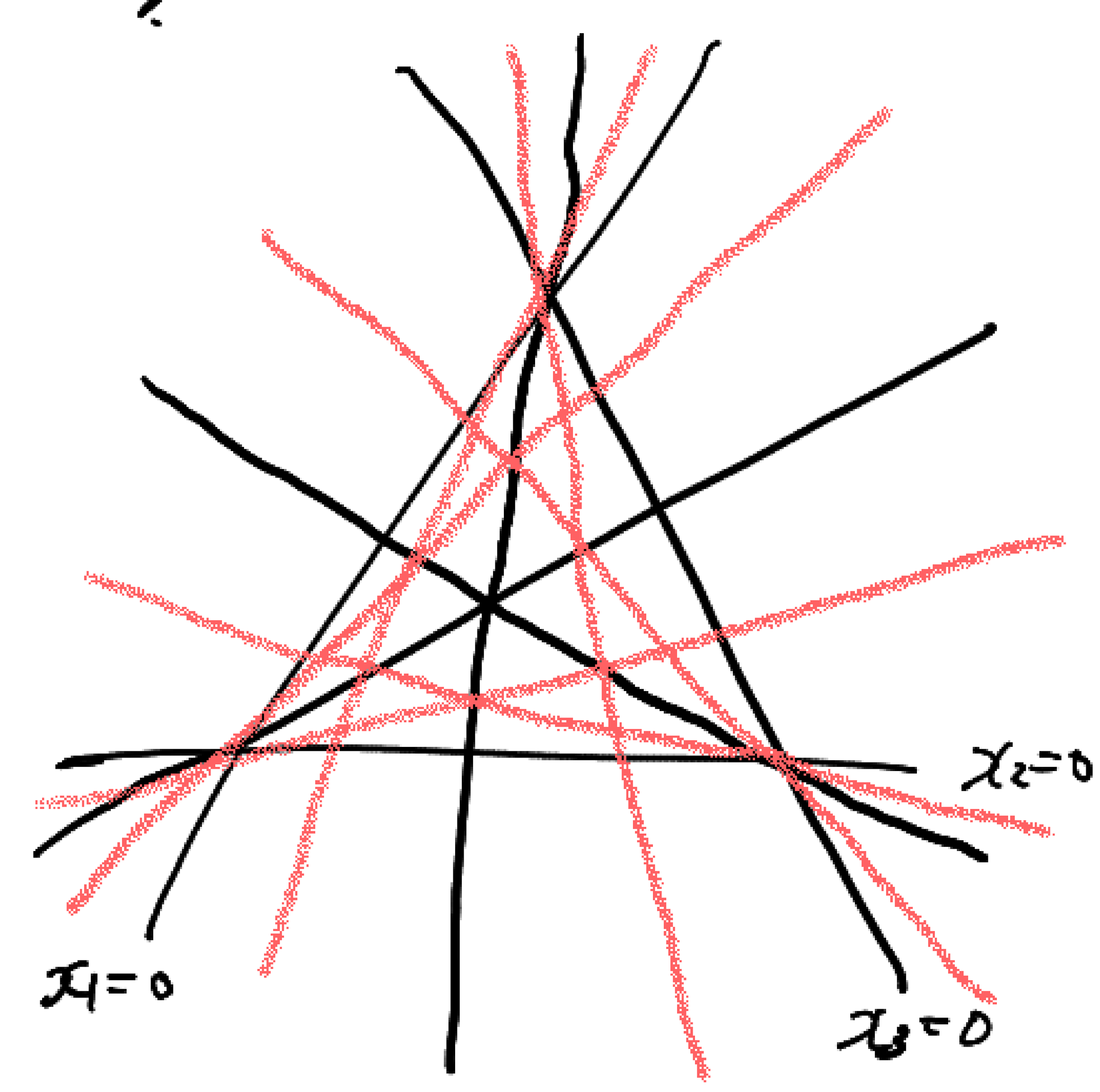
$B(\Gamma) : x_1 \cdots x_\ell \prod_{[i,j;k]\in E} (x_i - \varepsilon^k x_j)$

$K_\ell^{[-m,m]}$: edge set $= \left\{ [i,j;k] \;\middle|\; \begin{array}{l} 1\le i<j\le \ell \\ -m\le k\le m \end{array} \right\}$

$A(K_\ell^{[-m,m]}) = \mathrm{Cat}(\ell, m)$



$A(K_3^{[-1,1]}) = \mathrm{Cat}(3,1)$



$B(K_3^{[-1,1]})$

Thm (Deshpande-Menon-Sarkar)

$$\chi_{\mathcal{A}(\Gamma)}(t) = \chi_{\mathcal{B}(\Gamma)}(t+1)$$

Thm (Suyama-Torielli-T)

$c\mathcal{A}(\Gamma)$ は Γ の辺に沿って inductively free (divisionally)

$\Longleftrightarrow$ $\mathcal{B}(\Gamma)$ は ______________________

とくに, $\mathcal{B}(K_\ell^{[-m,m]})$, $\mathcal{B}(K_\ell^{[1-m,m]})$ は free.

Problem

- $c\mathcal{A}(\Gamma)$ is free $\overset{?}{\Longleftrightarrow}$ $\mathcal{B}(\Gamma)$ is free
- Why?
- $c\mathcal{A}(\Gamma)$, $\mathcal{B}(\Gamma)$ の freeness の特徴づけ (hard!!)

• $(A_{K_0}, 2m+1)$ (Bandlow-Musiker-Feigin)

$$\theta_k = \sum_{i,j=1}^{\ell} \left(\int_{x_i}^{x_j} t^k \prod_{p=1}^{\ell} (t-x_p)^m \, dt \right) \partial_i$$

• $\mathrm{Cat}(\ell, m) = \mathcal{A}(K_\ell^{[-m,m]})$ (Suyama-Yoshinaga)

$$\theta_k = \sum_{i,j=1}^{\ell} \left(\sum_{x_i}^{x_j} t^k \prod_{p=1}^{\ell} (t-x_p)^m \, \Delta t \right) \partial_i$$

• $\mathcal{B}(K_\ell^{[-m,m]})$ (Suyama-T)

$$\theta_k = \sum_{i,j=1}^{\ell} \left(x_i \int_{x_i}^{x_j} t^k \prod_{p=1}^{\ell} (t-x_p)_q^m \, d_q t \right) \partial_i$$

$$D_q(f(t)) = \frac{f(qt)-f(t)}{qt-t}$$

$\int d_q t$: D_q の逆演算

$$f(t)_q^m := f(t) f(q^{-1}t) \cdots f(q^{-m+1}t)$$

§q-deformation

$$\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell) := \{\ \mathbb{F}_q^\ell \text{ のすべての linear hyperplane}\ \}$$

$$Q = \prod_{i=1}^{\ell} \prod_{c_1, \ldots, c_{i-1} \in \mathbb{F}_q} (c_1 x_1 + \cdots + c_{i-1} x_{i-1} + x_i)$$

$$= \begin{vmatrix} x_1 & x_2 & \cdots & x_\ell \\ x_1^q & x_2^q & \cdots & x_\ell^q \\ \vdots & \vdots & & \vdots \\ x_1^{q^{\ell-1}} & x_2^{q^{\ell-1}} & \cdots & x_\ell^{q^{\ell-1}} \end{vmatrix} \in \mathbb{F}_q[x_1, \ldots, x_\ell]$$

Moore matrix

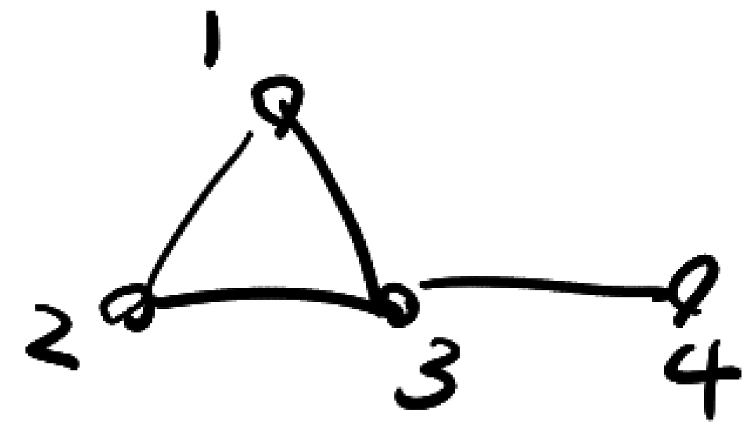
$$\chi_{\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)}(t) = (t-1)(t-q)(t-q^2)\cdots(t-q^{\ell-1})$$

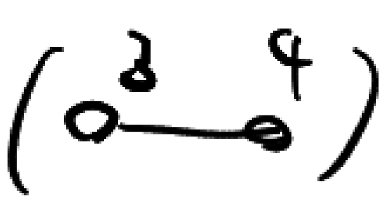
$$\chi_{\mathcal{A}(K_\ell)}(t) = t(t-1)(t-2)\cdots(t-\ell+1)$$

基底: $\theta_k = \sum_{i=1}^{\ell} x_i^{q^{k-1}} \partial_i$ for $D(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell))$

$\theta_k = \sum_{i=1}^{\ell} x_i^{k-1} \partial_i$ for $D(\mathcal{A}_{K_\ell})$

$$A_G^q := \bigcup_{\substack{\{i_1,\dots,i_r\} \\ \text{Gのクリーク}}} \left\{ \{a_{i_1}x_{i_1}+\cdots+a_{i_r}x_{i_r}=0\} \;\middle|\; (a_{i_1},\dots,a_{i_r})\in \mathbb{F}_q^r\setminus\{0\} \right\}$$

<u>Ex.</u> $G =$ 

A_G^q "=" Aall() $\cup$ Aall()

<u>Thm</u> (Nian-T-Uchiumi-Yoshinaga) TFAE

(1) A_G^q は supersolvable

(2) A_G^q は free

(3) G は chordal

~~基底~~ $$\theta_k = \sum_{i\in C_{\geq k}} \frac{\Delta_q(E_{<k}, x_i)}{\Delta_q(E_{<k})}\partial_i$$

<u>Rem</u> $q^k \leftrightarrow k$ の説明にはなっていない

Ex. $\chi_{C_\ell}(t) = (t-1)^\ell + (-1)^\ell (t-1)$

$$\chi_{\mathcal{A}^q_{C_\ell}}(t) = (t-q) + (-1)^\ell (q-1)^{\ell-1}(t-q)$$

$$\chi_G(k) = \#\{\kappa : [\ell] \to [k] \mid ij \in E \Rightarrow \kappa(i) \neq \kappa(j)\}$$

$$= \#\left\{(\kappa_1, \cdots, \kappa_\ell) \in [k]^\ell \;\middle|\; \begin{array}{l}\{i_1, \cdots, i_r\}\text{ がクリーク}\\ \Rightarrow \kappa_{i_1}, \cdots, \kappa_{i_r}\text{ は互いに相異なる}\end{array}\right\}$$

$$\chi_{\mathcal{A}^q_G}(q^k) = \#\{v \in (\mathbb{F}_q^k)^\ell \mid v \notin H \text{ for } {}^\forall H \in \mathcal{A}^q_G\}$$

$$= \#\left\{v \in (\mathbb{F}_q^k)^\ell \;\middle|\; \begin{array}{l}\{i_1, \cdots, i_r\}\text{ がクリーク}\\ \Rightarrow \{v_{i_1}, \cdots, v_{i_r}\}\text{ は }\mathbb{F}_q\text{ 上1次独立}\end{array}\right\}$$

$$\frac{\chi_{\mathcal{A}^q_G}(q^k)}{(q-1)^\ell} = \#\left\{\overline{v} \in \mathbb{P}(\mathbb{F}_q^k)^\ell \;\middle|\; \right\}$$

$\mathbb{P}(\mathbb{F}_q^k)$ は $[k]$ の q-analogue といわれている.

Question

$$\frac{\chi_{A_G^q}(q^k)}{(q-1)^\ell}$$ は q の多項式か?

$$\lim_{q\to 1} \frac{\chi_{A_G^q}(q^k)}{(q-1)^\ell} \overset{?}{=} \chi_G(k)$$

Ex.

$$\chi_{A_{C_\ell}^q}(t) = (t-q)^\ell + (-1)^\ell (q-1)^{\ell-1}(t-q)$$

$$\frac{\chi_{A_{C_\ell}^q}(q^k)}{(q-1)^\ell} = \frac{(q^k-q)^\ell}{(q-1)^\ell} + (-1)^\ell \frac{q^k-q}{q-1}$$

$$= q^\ell \left(\frac{q^{k-1}-1}{q-1}\right)^\ell + (-1)^\ell q \left(\frac{q^{k-1}-1}{q-1}\right)$$

$$\longrightarrow (k-1)^\ell + (-1)^\ell (k-1) = \chi_{C_\ell}(k)$$

Prop G : K_3-free

Then $\chi_{A_G^q}(t) = (q-1)^\ell \chi_G\left(\frac{t-1}{q-1}\right)$

$\because$ $$\frac{\chi_{A_G^q}(q^k)}{(q-1)^\ell} = \#\left\{ \bar{v} \in \mathbb{P}(\mathbb{F}_q^k)^\ell \;\middle|\; \bar{v}_i \neq \bar{v}_j \text{ if } ij \in E \right\}$$

$$= \chi_G(\#\mathbb{P}(\mathbb{F}_q^k)) = \chi_G\left(\frac{q^k-1}{q-1}\right)$$ $\square$

Thm (Nian-T-Uchiumi-Yoshinaga)

$$\frac{\chi_{\mathcal{A}_G^q}(q^k)}{(q-1)^{\ell}} \equiv \chi_G(k) \pmod{q-1}$$

とくに、qが十分に大きければ、$\chi_{\mathcal{A}_G^q}(t)$から$\chi_G(t)$が分かる。

Ex $\chi^q_{C_4}(t) = (t-q)^4 + (-1)^4(q-1)^3(t-q)$

$= t^4 - a_3(q)t^3 + a_2(q)t^2 - a_1(q)t + a_0(q)$

$(a_0, a_1, a_2, a_3, 1)$ は log-concave ($\forall q$: 素数べき)

$q=1$ で Taylor 展開してみる。

$a_4 = 1$

$a_3 = 4 + 4(q-1)$

$a_2 = 6 + 12(q-1) + 6(q-1)^2$

$a_1 = 4 + 12(q-1) + 12(q-1)^2 + 3(q-1)^3$

$a_0 = 1 + 4(q-1) + 6(q-1)^2 + 3(q-1)^3$

$$a_1^2 - a_0 a_2 = 10 + 60(q-1) + 150(q-1)^2 + 198(q-1)^3 + 144(q-1)^4 + 54(q-1)^5 + 9(q-1)^6$$

$$a_2^2 - a_1 a_3 = 20 + 80(q-1) + 120(q-1)^2 + 84(q-1)^3 + 24(q-1)^4$$

$$a_3^2 - a_2 a_4 = 10 + 20(q-1) + 10(q-1)^2$$

Question

$$\chi_{A_G^q}(t) = \sum_{i=0}^{\ell} (-1)^{\ell-i} a_i(q) t^i$$

- $a_i(q)$はqの多項式か？
- $a_i(q)$を$q=1$でTaylor展開すると、係数は正か？
- $a_i(q)^2 - a_{i-1}(q)a_{i+1}(q)$を$q=1$で ______ ？