可換代数と組合せ論

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Contents

1 Introduction

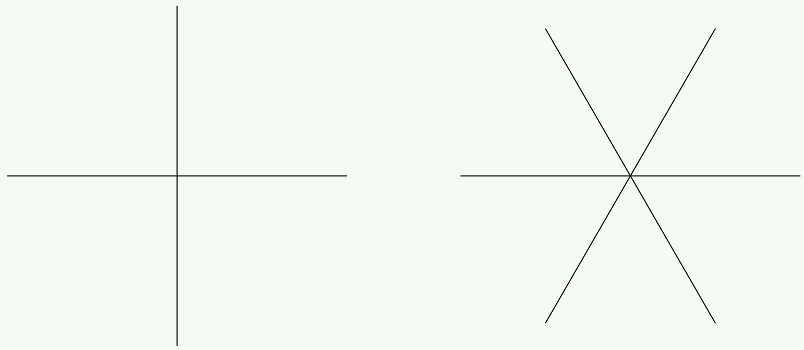
1.1 Equiangular Lines

We introduce definitions and notation for equiangular lines.

Definition 1. A set of lines in $\mathbb{R}^d$ is called equiangular if the angle between any two lines is the same.

Definition 2. Let $d \in \mathbb{Z}_{\geq 1}$. Denote by $N(d)$ the maximum number of equiangular lines in $\mathbb{R}^d$. For a common angle $\arccos(1/\alpha)$, denote by $N_\alpha(d)$ the maximum number of equiangular lines in $\mathbb{R}^d$ with common angle $\arccos(1/\alpha)$.

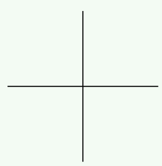
Example 3. The following are examples of equiangular lines in $\mathbb{R}^2$.



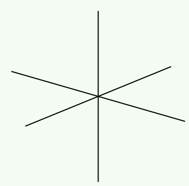
Definition 4. A set of equiangular lines is called <u>maximal</u> if there is no set of equiangular lines in $\langle U \rangle_{\mathbb{R}}$ properly containing it.

Definition 5. A set of equiangular lines is called <u>strongly maximal</u> if there is no set of equiangular lines properly containing it.

Example 6. The following are examples of maximal and strongly maximal equiangular lines in $\mathbb{R}^2$.

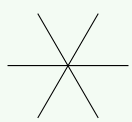


$\subset \mathbb{R}^2 \quad \longrightarrow$



$\subset \mathbb{R}^3$

maximal
(not strongly maximal)



$\subset \mathbb{R}^2 \quad \longrightarrow$ (does not exist)

strongly maximal

1.2 Motivation

Problems concerning how points or geometric objects can be arranged in space are among the fundamental subjects of study in discrete geometry.

- Sphere packing problem: How densely can congruent spheres be packed in space?
- Kissing number problem: How many congruent spheres can be arranged so that each touches a given sphere?

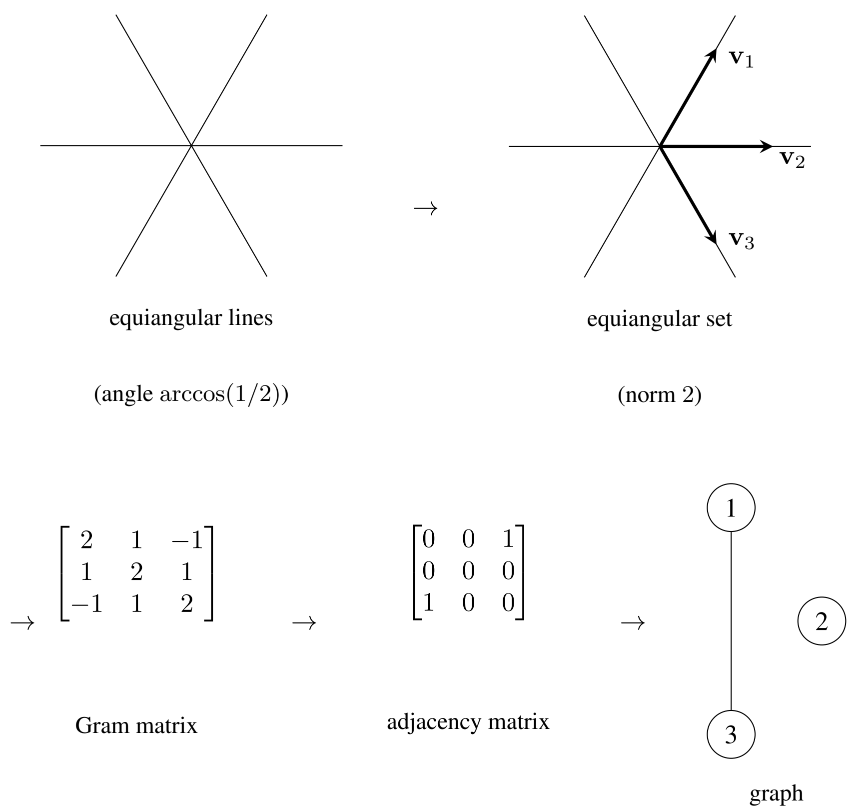
1.3 Terminology: Equiangular Sets

Definition 7. An equiangular set of norm $\alpha \in \mathbb{R}_{>0}$ is a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^N$ such that

$$(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} \alpha & \text{if } i = j, \\ -1, 1 & \text{otherwise.} \end{cases}$$

As shown in Figure 1.3, an affine equiangular set of norm α is induced by an equiangular set of lines with common angle $\arccos(1/\alpha)$. Since vectors $\mathbf{v}_i$ in an affine equiangular set of norm α can be switched to $-\mathbf{v}_i$, the resulting set is

sets-graphs



said to be switching equivalent to the original set.

A Seidel matrix is a symmetric matrix with zero diagonal and ± 1 off-diagonal entries. An affine equiangular set of norm α has the Gram matrix

$$\alpha I + S$$

for some Seidel matrix S. Furthermore, there exists a graph G such that

$$S := S(G) = J - I - 2A(G),$$

where $A(G)$ is the adjacency matrix of G. See the above figure for an example. If an equiangular set is switched, then the corresponding graph is switched, in the sense of the following definition.

Definition 8. A graph G is said to be switching equivalent to a graph H if there exists $X \subset V(G)$ such that

$$V(G) = V(H), \quad E(H) = E(G) \triangle \{\{u, v\} : u \in X, v \in V(G) \setminus X\}.$$

Determining whether two graphs are switching equivalent is reduced to checking isomorphism between two associated graphs.

1.4 Bounds and Asymptotic Behavior

We first recall general upper bounds for $N(d)$ and $N_\alpha(d)$.

Theorem 9 (Absolute bound lemmens1973 [4, Theorem 3.5])**.** *For every integer $d \geq 1$,*

$$N(d) \leq \frac{d(d+1)}{2}.$$

If this is attained, then the common angle is $\arccos(1/\sqrt{d+2})$.

Only the following examples are known to attain this bound.

d	α	
2	2	Hexagon
3	$\sqrt{5}$	Icosahedron
7	3	Root lattice E_8
23	5	Leech lattice

To prove the absolute bound, we use the following fact. Let

$$\mathrm{Sym}_d(\mathbb{R}) := \{P \in M_d(\mathbb{R}) : P^T = P\}.$$

In general, for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, letting $X := \mathbf{u}\mathbf{u}^T$ and $Y := \mathbf{v}\mathbf{v}^T$, we have

$$\mathrm{tr}(XY) = \mathrm{tr}(\mathbf{u}\mathbf{u}^T\mathbf{v}\mathbf{v}^T) = (\mathbf{u}, \mathbf{v})\,\mathrm{tr}(\mathbf{u}\mathbf{v}^\top) = (\mathbf{u}, \mathbf{v})^2.$$

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an equiangular set of norm α in $\mathbb{R}^d$. Let

$$X_i := \mathbf{u}_i\mathbf{u}_i^T \in \mathrm{Sym}_d(\mathbb{R}) \qquad (i = 1, \ldots, n).$$

Then, $X_1, \ldots, X_n$ are linearly independent in $\mathrm{Sym}_d(\mathbb{R})$, since

$$\begin{aligned}
&\sum_{i=1}^n c_i X_i = 0 \\
&\implies 0 = \mathrm{tr}(\left(\sum_{i=1}^n c_i X_i\right)^2) = \sum_{i,j=1}^n c_i c_j\,\mathrm{tr}(X_i X_j) = \sum_i c_i^2 \alpha^2 + \sum_{i \neq j} c_i c_j = (\alpha^2 - 1)\sum_i c_i^2 + \left(\sum_i c_i\right)^2 \\
&\implies c_1 = \cdots = c_n = 0.
\end{aligned}$$

Therefore,

$$n = \dim \mathrm{Span}_{\mathbb{R}}\{X_1, \ldots, X_n\} \leq \dim \mathrm{Sym}_d(\mathbb{R}) = \frac{d(d+1)}{2}.$$

If equality holds, then $I, X_1, \ldots, X_n$ are linearly dependent. This implies that the common angle is $\arccos(1/\sqrt{d+2})$ after a straightforward calculation. $\square$

Theorem 10 (Relative bound lemmens1973 [4, Theorem 3.6])**.** *Let* $\alpha > 1$ *and* $d \in \mathbb{Z}_{\geq 1}$*. If* $d < \alpha^2$*, then*

$$N_\alpha(d) \leq \frac{d(\alpha^2 - 1)}{\alpha^2 - d}.$$

If a set of equiangular lines attains the relative bound, then its Seidel matrix has only two distinct eigenvalues, that is, it corresponds to a two-graph. We next record a general lower bound for $N(d)$.

Theorem 11 (Greaves–Koolen–Munemasa–Szöllősi GHAF2016 [2, Corollary 2.8])**.** *For every integer* $d \geq 1$*,*

$$N(d) \geq \frac{32d^2 + 328d + 296}{1089}.$$

Hence the unrestricted problem has quadratic growth.

Corollary 12.

$$N(d) = \Theta(d^2).$$

We now turn to asymptotic results for fixed common angles.

Theorem 13. *If* $N_\alpha(d) \geq 2d + 1$*, then* α *is an odd integer.*

Theorem 14 (Lemmens–Seidel lemmens1973 [4, Theorem 3.6])**.** *For every integer* $d \geq 15$*,*

$$N_3(d) = 2(d - 1).$$

The asymptotic formula for $N_5(d)$ was conjectured by Lemmens and Seidel and established in later works.

Theorem 15. *For all sufficiently large integers* d*,*

$$N_5(d) = \left\lfloor \frac{3(d-1)}{2} \right\rfloor.$$

As a generalization of the above results, we recall the theorem of Jiang–Tidor–Yao–Zhang–Zhao.

Corollary 16 (Jiang–Tidor–Yao–Zhang–Zhao Zhao2021 [3, Corollary 1.3])**.** *Let* $\alpha \geq 3$ *be an odd integer. Then*

$$N_\alpha(d) = \left\lfloor \frac{(\alpha+1)(d-1)}{\alpha - 1} \right\rfloor$$

for all sufficiently large d*.*

1.5 Low Dimensions

PC 列挙できる　Pillar法 + (rel.)　スペクトラルグラフ + PC

d	2	3	4	5	6	7–14	15	16	
$N(d)$	3	6	6	10	16	28	36	40	
d	17	18	19	20	21	22	23–41	42	43
$N(d)$	48	57–59	72–74	90–94	126	176	276	276–288	344

PC　スペクトラルグラフ + PC　上の次元から削る　(+rel)　Pillar、禁止グラフ　SDP

1.6 Strong Maximality

Theorem 17. *If a set of equiangular lines attains the absolute bound, then it is strongly maximal.*

Furthermore, if $\alpha = 2, \sqrt{5}, 3$, then there exists a unique strongly maximal set of equiangular lines with common angle $\arccos(1/\alpha)$. However, if $\alpha = 5$, then an analogous statement does not hold: there are many strongly maximal sets of equiangular lines with common angle $\arccos(1/5)$.

Question 18. For any odd integer α, is there a strongly maximal set of equiangular lines with common angle $\arccos(1/\alpha)$?

The following facts are known about strongly maximal sets of equiangular lines.

- $\bar{K}_n$ is strongly maximal iff n is odd. (The angle is $\arccos(1/(n-1))$).
- $K_n + E_m$ $(n, m \geq 2)$ is strongly maximal iff $\alpha \in 2\mathbb{Z}$ when $\alpha \in \mathbb{Z}$, and ....
- Let $\alpha < 3$. If there exists a set of equiangular lines with common angle $\arccos(1/\alpha)$, then there exists a strongly maximal set of equiangular lines with common angle $\arccos(1/\alpha)$.
- If a Seidel matrix has only two distinct eigenvalues and one is not an integer, then it is strongly maximal. For a prime power congruent to 1 modulo 4, the direct sum of the Paley graph of order q and K_1 has eigenvalues $\pm\sqrt{q}$.

2 Affine Equiangular Sets

dfn:aes

Definition 19. Let $\beta \in \mathbb{R}_{\geq 1}$. Let $\mathbf{r}$ be a root in $\mathbb{R}^N$, where $N \in \mathbb{Z}_{\geq 1}$. The set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in $\mathbb{R}^N$ is called an affine equiangular set with norm β with respect to $\mathbf{r}$ if

(i) $(\mathbf{u}_1, \mathbf{r}) = \cdots = (\mathbf{u}_n, \mathbf{r}) = 1$,

(ii) $(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases} \beta & \text{if } i = j, \\ 0, 1 & \text{otherwise.} \end{cases}$

The root $\mathbf{r}$ is called the switching root of the affine equiangular set. For $\mathbf{u} \in \omega$ with $\mathbf{r} - \mathbf{u} \notin \omega$, the pair

$$[\mathbf{u}]_{\mathbf{r}} := \{\mathbf{u}, \mathbf{r} - \mathbf{u}\}.$$

is called a switching pair.

Definition 20. For a vector $\mathbf{u}$ and a root $\mathbf{r}$, define a map $\mathcal{L}_{\mathbf{r}}$ by

$$\mathcal{L}_{\mathbf{r}}(\mathbf{u}) := \frac{2\mathbf{u} - \mathbf{r}}{\sqrt{2}} \quad \text{if } (\mathbf{u}, \mathbf{r}) = 1.$$

For short, we write $\mathcal{L}_{\mathbf{r}}(\mathbf{u})$ as $\mathcal{L}(\mathbf{u})$. Also, for a set ω of vectors, define $\mathbb{R}\mathcal{L}(\omega) := \{\mathbb{R}\mathcal{L}(\mathbf{u}) : \mathbf{u} \in \omega\}$.

Note 21. For an affine equiangular set $\omega = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ with norm α with respect to a switching root $\mathbf{r}$, we have

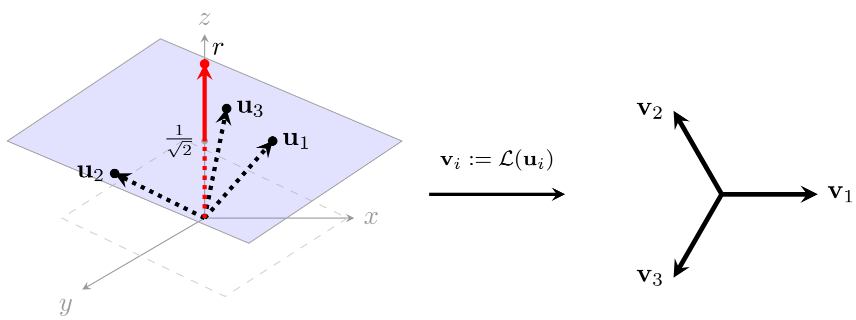
$$(\mathcal{L}(\mathbf{u}_i), \mathcal{L}(\mathbf{u}_j)) = \begin{cases} 2\alpha - 1 & \text{if } i = j, \\ -1, 1 & \text{otherwise.} \end{cases}$$

Furthermore, the lines

$$\{\mathbb{R}\mathcal{L}(\mathbf{u}) : \mathbf{u} \in \omega\} \subset \langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle \cap \mathbf{r}^{\perp}$$

are equiangular with common angle $\arccos(1/(2\alpha - 1))$. We say that this set of equiangular lines is induced by ω. Any set of equiangular lines with common angle $\arccos(1/(2\alpha - 1))$ is induced by some affine equiangular set with norm α.

Affine equiangular sets of norm $s \overset{1:1}{\longleftrightarrow}$ Equiangular set of norm $2s - 1$.



Affine equiangular set

$$(\mathbf{u}_i, \mathbf{u}_j) = \begin{cases} s & \text{if } i = j, \\ 0, 1 & \text{o.w.} \end{cases}$$

Equiangular set

$$(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 2\alpha - 1 & \text{if } i = j, \\ -1, 1 & \text{o.w.} \end{cases}$$

itching_equiv

Definition 22 (Switching)**.** Two affine equiangular sets ω and ω' with respect to a common switching root $\mathbf{r}$ are said to be switching equivalent if there exists a bijection $g : \omega \to \omega'$ such that

$$g(\mathbf{u}) \in \{\mathbf{u}, \mathbf{r} - \mathbf{u}\}$$

for every $\mathbf{u} \in \omega$. Denote by $\sim_{\text{sw}}$ the switching equivalence, that is,

$$\omega \sim_{\text{sw}} \omega'$$

if ω and ω' are switching equivalent. Define

$$[\omega] := \{\omega' : \omega' \sim_{\text{sw}} \omega\}.$$

Note 23. Since

$$(\mathbf{u}, \mathbf{r} - \mathbf{u}) = 1 - \beta$$

and $\beta \geq 1$, we have

$$\mathbf{u} \in \omega \implies \mathbf{r} - \mathbf{u} \notin \omega.$$

Also, since

$$\mathcal{L}(\mathbf{r} - \mathbf{u}) = \frac{2(\mathbf{r} - \mathbf{u}) - \mathbf{r}}{\sqrt{2}} = \frac{\mathbf{r} - 2\mathbf{u}}{\sqrt{2}} = -\frac{2\mathbf{u} - \mathbf{r}}{\sqrt{2}} = -\mathcal{L}(\mathbf{u}),$$

we have

$$\mathbb{R}\mathcal{L}(\mathbf{u}) = \mathbb{R}\mathcal{L}(\mathbf{r} - \mathbf{u}).$$

For two affine equiangular sets ω and ω' with respect to a common switching root $\mathbf{r}$,

$$\omega \sim_{\mathrm{sw}} \omega' \implies \mathbb{R}\mathcal{L}(\omega) = \mathbb{R}\mathcal{L}(\omega').$$

Definition 24 (Isometric)**.** Affine equiangular sets ω with respect to a switching root $\mathbf{r}$ and ω' with respect to a switching root $\mathbf{r}'$ are isometric, denoted $\omega \simeq \omega'$, if there exists an affine equiangular set ω'' switching equivalent to ω' and a bijective isometry $g : \omega \cup \{\mathbf{r}\} \to \omega'' \cup \{\mathbf{r}'\}$ such that $g(\mathbf{r}) = \mathbf{r}'$.

Note 25. For two affine equiangular sets ω and ω' with respect to switching roots $\mathbf{r}$ and $\mathbf{r}'$, respectively, we have

$$\begin{aligned} \omega \sim \omega' &\implies \omega \simeq \omega', \\ \omega \simeq \omega' &\implies \mathbb{R}\mathcal{L}(\omega) \simeq \mathbb{R}\mathcal{L}(\omega'). \end{aligned}$$

3 On Root Lattices

Definition 26. A lattice is a $\mathbb{Z}$-span of a $\mathbb{R}$-linearly independent (finite) set of vectors in $\mathbb{R}^N$. A lattice L is said to be integral if $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}$ for all $\mathbf{u}, \mathbf{v} \in \mathsf{L}$. A lattice is called a root lattice if it is generated by roots. Define

$$\mathsf{R}(\mathsf{L}) := \{\mathbf{r} \in \mathsf{L} : (\mathbf{r}, \mathbf{r}) = 2\}.$$

It is known that the irreducible root lattices are enumerated up to bijective isometry as follows (cf. Lattices2013 [1, Theorem 1.2]).

$$\begin{aligned} \mathsf{A}_n &:= \{\mathbf{v} \in \mathbb{Z}^{n+1} : (\mathbf{v}, \mathbf{j}) = 0\} \quad (n \in \mathbb{Z}_{\geq 1}), \\ \mathsf{D}_n &:= \{\mathbf{v} \in \mathbb{Z}^{n} : (\mathbf{v}, \mathbf{j}) \in 2\mathbb{Z}\} \quad (n \in \mathbb{Z}_{\geq 4}), \\ \mathsf{E}_8 &:= \mathsf{D}_8 \sqcup (\mathbf{j}/2 + \mathsf{D}_8), \\ \mathsf{E}_7 &:= \{\mathbf{v} \in \mathsf{E}_8 : (\mathbf{v}, \mathsf{e}_1 - \mathsf{e}_2) = 0\}, \\ \mathsf{E}_6 &:= \{\mathbf{v} \in \mathsf{E}_8 : (\mathbf{v}, \mathsf{e}_1 - \mathsf{e}_2) = (\mathbf{v}, \mathsf{e}_2 - \mathsf{e}_3) = 0\}. \end{aligned}$$

The following is well known.

$$\begin{aligned} &\mathsf{D}_4 \subset \mathsf{D}_5 \subset \cdots, \mathsf{E}_6 \subset \mathsf{E}_7 \subset \mathsf{E}_8, \\ &\mathsf{D}_6 \not\subset \mathsf{E}_6, \mathsf{D}_7 \not\subset \mathsf{E}_7, \mathsf{D}_8 \subset \mathsf{E}_8, \\ &\mathsf{E}_n \not\subset \mathsf{D}_{n'} \text{ for } n \text{ and } n'. \end{aligned}$$

The Weyl group $W(\mathsf{L})$ of a root lattice L is the subgroup of $\mathrm{Aut}(\mathsf{L})$ generated by all the reflections $s_{\mathbf{r}}$ corresponding to roots $\mathbf{r}$, where $s_{\mathbf{r}}(\mathbf{x}) = \mathbf{x} - 2\mathbf{r}(\mathbf{x},\mathbf{r})/(\mathbf{r},\mathbf{r}) = \mathbf{x} - \mathbf{r}(\mathbf{x},\mathbf{r})$.

4 Affine Equiangular Sets of Norm 2

The sets of equiangular lines with common angle $\arccos(1/3)$ are described by root lattices.

4.1 Maximality

We determine the maximal sets of equiangular lines with common angle $\arccos(1/3)$ using root lattices. An affine

table:N3(d)

Table 1: Maximal sets

d	4	5	6	7	≥ 8
$\#$	6	$8, 10$	$12, 16$	28	$2(d-1)$
	$L(2,3)$	$L(2,4), L(K_5)$	$L(2,5), L(K_6) + K_1$	$L(K_8)$	$L(K_{2,d-2})$
L	D_5	$\mathsf{D}_6, \mathsf{E}_6$	$\mathsf{D}_7, \mathsf{E}_7$	E_8	D_{d-1}

equiangular set ω with norm 2, together with its switching root $\mathbf{r}$, generates an irreducible root lattice $\mathsf{L} := \langle \omega \cup \{\mathbf{r}\} \rangle$. Then, we have

$$\omega \subset \{\mathbf{u} \in \mathsf{R}(\mathsf{L}) : (\mathbf{u},\mathbf{r}) = 1\}\,.$$

Conversely, we can construct an affine equiangular set with norm 2 from an irreducible root lattice.

Definition 27. Let L be an irreducible root lattice, and let $\mathbf{r} \in \mathsf{L}$ be a root.

$$N_{\mathbf{r}}(\mathsf{L}) := \{\mathbf{u} \in \mathsf{R}(\mathsf{L}) : (\mathbf{u},\mathbf{r}) = 1\}.$$

Then, $\mathbf{r} \in \mathsf{R}(\mathsf{L}) \implies \mathbf{r} - \mathbf{u} \in \mathsf{R}(\mathsf{L})$ and

$$[\omega] \overset{1:1}{\longleftrightarrow} \{[\mathbf{u}]_{\mathbf{r}} : \mathbf{u} \in \omega\} \subset N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}\,.$$

Lemma 28. *Let L be an irreducible root lattice, and let $\mathbf{r} \in \mathsf{L}$ be a root. Then, $N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}$ is the switching class of an affine equiangular set with norm 2 with respect to $\mathbf{r}$.*

Proof. Let $[\mathbf{u}]_{\mathbf{r}} \neq [\mathbf{v}]_{\mathbf{r}} \in N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}$. We show that $(\mathbf{u},\mathbf{v}) \in \{0,1\}$. Indeed, it suffices to show that $(\mathbf{u},\mathbf{v}) \neq -2, -1$

$$(\mathbf{u},\mathbf{v}) = -2 \implies 1 = (\mathbf{u},\mathbf{r}) = -(\mathbf{v},\mathbf{r}) = -1.$$

This is a contradiction. Also,

$$\begin{aligned} &(\mathbf{u},\mathbf{v}) = -1 \\ &\implies \mathbf{u}+\mathbf{v} \in \mathsf{R}(\mathsf{L}) \text{ and } (\mathbf{u}+\mathbf{v},\mathbf{r}) = 2 \\ &\implies \mathbf{u}+\mathbf{v} = \mathbf{r} \\ &\implies \mathbf{u} \sim_{\mathrm{sw}} \mathbf{v}. \end{aligned}$$

This is a contradiction. Therefore, $(\mathbf{u},\mathbf{v}) \in \{0,1\}$. $\square$

em:transitive **Lemma 29** (Lattices2013 [1, Lemma 1.10])**.** *The Weyl group of an irreducible root lattice acts transitively on the set of its roots.*

Corollary 30. *The switching classes of maximal affine equiangular sets are $N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}$ ($\mathsf{L}=\mathsf{D}_n$ ($n\neq 8$), $\mathsf{E}_6,\mathsf{E}_7,\mathsf{E}_8$).*

Proof. The candidate switching classes of maximal affine equiangular sets are $N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}$, where L is an irreducible root lattice. By Lemma 29, this does not depend on the choice of $\mathbf{r}$. If

$$\mathsf{L}\subset\mathsf{M}\implies N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}\subset N_{\mathbf{r}}(\mathsf{M})/\sim_{\mathrm{sw}},$$

then, by the containment relations among irreducible root lattices, the maximal ones are $N_{\mathbf{r}}(\mathsf{L})/\sim_{\mathrm{sw}}$ where L is a maximal irreducible root lattice, that is, D_n ($n\neq 8$), $\mathsf{E}_6,\mathsf{E}_7,\mathsf{E}_8$. □

Example 31 ($N_{\{}\mathsf{e}_1+\mathsf{e}_2\}(\mathsf{D}_n)/\sim_{\mathrm{sw}}$)**.** Let $\mathbf{r}=\mathsf{e}_1+\mathsf{e}_2$. Recall that

$$\mathsf{D}_n=\{\mathbf{v}\in\mathbb{Z}^n:(\mathbf{v},\mathbf{j})\in 2\mathbb{Z}\}.$$

Hence,

$$N_{\mathsf{e}_1+\mathsf{e}_2}(\mathsf{D}_n)=\{\mathsf{e}_i+\mathsf{e}_j: i=1,2, j=3,\dots,n\}.$$

This implies

$$N_{\mathsf{e}_1+\mathsf{e}_2}(\mathsf{D}_n)/\sim_{\mathrm{sw}}=\{[\mathsf{e}_1+\mathsf{e}_j]_{\mathbf{r}},[\mathsf{e}_2+\mathsf{e}_j]_{\mathbf{r}}: j=3,\dots,n\}.$$

Then, $\mathsf{e}_i+\mathsf{e}_j$ corresponds to the edge $\{i,j\}$ of $K_{2,n}$ with partite sets $\{1,2\}$ and $\{3,\dots,n\}$. Hence, the Gram matrix of $\mathsf{e}_i+\mathsf{e}_j$ ($i=1,2$ and $j=3,\dots,n$) is

$$2I+A(L(K_{2,n-2})).$$

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